Strategies for Proof Compression in Advanced Calculus

A Way to Help Students Internalize Complex Proofs More Efficiently

Scott Beaver - Western Oregon University

MAA Pacific Northwest Section Annual Meeting
April 13, 2007

What are the aspects of Advanced Calculus which lead to student difficulties?

- Almost all of the problems are proofs, so ideally logic should be internalized *prior* to taking the course
  - Most undergraduate students realize that they are not yet fluent in logic
- The totality of the proofs in the course encompass a seemingly endless variety of approaches and techniques
- Many of the proofs are abstract
- Many of the proofs are not short
- Many of the proofs are motivated quite sensibly, but understanding how a given proof is motivated may come only after much study

These issues translate into the following common challenges

- Figuring out how to begin a proof
- Recalling how to proceed in a previously-worked proof
- Finding sufficient time to study for exams *because they try to read through/memorize all of each proof*

To address these challenges, I require that, for problems of more than minimal complexity, students write a (scored) strategy, containing no computational or algebraic details, above the problem solution

This is often done after the fact, but the idea is that students will develop the habit of thinking strategically first

I want students who do not possess strong aptitude for Advanced Calculus to adopt a two-phase process

- develop a strategy
- implement the strategy using any sufficient tactics

The goal is fourfold

- to help students study for exams without necessarily having to read every detail of every problem solution
- to manifest similarities and differences between problem solutions and hence help students gain a broader perspective
- to help students differentiate between the strategy, and algebraic and computational techniques they employ in proofs
- to eventually train students to develop clear, strategy-based lines of thought

What is a strategy?

- The idea is the same as that of data compression for storage - retain only the “essentials” and allow the computer (student’s brain) to fill in the details
- This will necessarily lead to varying degrees of detail in sufficient strategies for different students

What should students be writing, or not writing, in a good strategy?

- Cite any theorems used, but don't state them fully
- Cite definitions if relatively recent, but don't state them fully
- Explain why the hypotheses of those theorems are satisfied, if not obvious
- If you need to peek at the complete proof to understand the strategy, the strategy is incomplete
- Avoid computational details if at all possible
- Use language instead of symbols if you need to

Problem

Show that between any two unequal real numbers, there are an infinite number of rational numbers.

Strategy

Fix $a, b \in \mathbb{R}$ with $a < b$. $\mathbb{Q}$ dense in $\mathbb{R}$ yields existence of a rational r in (a, b). ß BWOC that $|\mathbb{Q} \cap (a, b)| = N < \infty$. Use density again to contradict that.

Problem

Show that if $a_n \geq 0$ and $\sum a_n$ converges, then if $p > 1$, then $\sum a_n^p$ converges.

Strategy

$\sum a_n$ converges $\Rightarrow$ $a_n \to 0$. So eventually, all the a_n's are in $[0, 1)$. Use Comparison Test with $\sum a_n^p$ and $\sum a_n$ on this tail, and note that the head of a series is irrelevant for convergence purposes.

Problem

Let (f_n) be a sequence of continuous functions on $[a, b]$ and S $\mathrm{f}_\mathrm{n} \xrightarrow{\mathrm{u}} \mathrm{f}$. *Prove that if $(x_n) \in [a, b]^\infty$, with $x_n \to x$, then $f_n(x_n) \to f(x)$.*

Strategy

Note that $x \in [a, b]$. Use Add-and-Subtract and the Triangle Inequality; make each term of $|f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$ $\frac{\varepsilon}{2}$-small. By $f_n \xrightarrow{u} f$, we can shrink the first term irrespective of the x_n's. Also $f_n \xrightarrow{u} f \Rightarrow f$ continuous by Theorem 24.3. Thus the second term is small, if $|x_n - x|$ is small, which eventually holds since $x_n \to x$. Use $N = \max\{N_1, N_2\}$.

I don't ask for a strategy on any problem in which I ask for a complete proof; too time-consuming

Exams can be made more efficient by including two complementary kinds of problems

- Find a strategy only, no complete proof - limited applicability
- Given a strategy,
  - fill in the details, or
  - show why the strategy won't work

What some students said:

- "I felt that by writing strategies I gained a better understanding on how to write out proofs."
- "I think it is a good idea and should be used in all upper division math courses."
- "I think that this was a really good way to help the student learn the material and not just regurgitate the information that was presented to them."
- "Reading a few English sentences in my own words is less confusing than trying to figure out what all the notation, symbols, and variables are for."
- "I think that this is a good tool for learning and look forward to using it myself."

- Can be used in traditional lecture courses or in discovery-based courses
- Calculus III - Sequences and Series; students typically can't give rigorous proofs, but are expected to use some elementary logic
- Multivariable Calculus; calculations often require several disparate steps
- Any proof-heavy course like Group Theory or Ring Theory, Topology
- Applicable to Complex Analysis, to proofs and many of the computations therein