

The Natural Role of the Sequences and Series Calculus Course

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MAA MathFest Session:
Getting Students Involved in Writing Proofs

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Students' (and Citizens') Difficulties with Logic

Junior and senior math major students should have little trouble generating strategies for, and completing, proofs of reasonable complexity

In fact, many students find it difficult to have internalized applied logical reasoning even after several proof-intensive courses

As for the logical abilities of our citizenry in general, there's no pretending that society wouldn't benefit from better training in logic

Hey, it saved the planet Vulcan, right?

Humbly Offered Partial Solutions

How do we, as a society, improve our chances of internalizing this skill earlier? **Age-appropriate formal logic for four weeks in every grade through K-12! (There is some movement in this direction, thankfully)**

That's a much bigger fight; for now I'll apply that hypothesis to my own children (ages $3\frac{221}{365}$ and $6\frac{126}{365}$) and see what happens

As for the “Academy” - how do we, as college instructors, improve their chances of internalizing this skill as soon as possible?

- Early bridge courses
- More emphasis on proofs/logic in sophomore courses
- *Structure Sequences and Series as a proofs course*

The Sequences and Series Course at Western Oregon University

- WOU is on a quarter-term schedule, so Sequences and Series is Calculus III
- Historically a teacher training school, WOU does not service any engineering program
- The S & S demographic: 75 % math majors, 15 % math minors, 10 % Computer Science and others
- Typically taken during Spring term, after differential (Fall term) and integral (Winter term) calculus

The Calculus Sequence

Interestingly, almost every textbook follows this paradigm:

- 1 Differential
- 2 Integral
- 3 Sequences and Series
- 4 Multivariate

Pedagogically, I find this quite curious

The Typical Problem Type in Each Course

- 1 Differential - **Algorithmic** (e.g., finding the slope at a point or the derivative function)
- 2 Integral - **Algorithmic** (e.g., finding the area under the curve or a family of antiderivatives)
- 3 Sequences and Series - **Logical** (e.g. arguing that particular sequences or series converge or fail to do so) and to a much lesser extent algorithmic (e.g., finding the Taylor series for a function about a point and the associated interval of convergence)
- 4 Multivariate - **Algorithmic** (e.g., finding vector functions, equations of lines, planes; finding partial derivatives, volumes, surface areas; optimization problems)

The Transition from Algorithmic to Logical

Math students are typically taught to solve problems algorithmically in math courses through integral calculus

Then, suddenly and out of nowhere in Sequences and Series (hereafter S & S), “justify your answer” plays a truly integral role (no pun) (OK, pun)

After that, it's back to the comfort zone for multivariate differentiation and integration

During the sophomore year or junior year, we begin to demand more of their logical faculties, perhaps after a bridge course

A Way to Ease the Transition, Right Under our Noses

Sequences and Series *is* a Proofs Course!

The vast majority of types of problems students encounter in S & S require argument and justification

At WOU, it helps that the most populated section of S & S is offered during the same term as our bridge course

Is This in Your Calculus Text?

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n} + 1} = \frac{1}{\lim_{n \rightarrow \infty} \frac{1}{n} + 1} = \frac{1}{0+1} = 1.$$

- 1 How do we know the LHS exists?
- 2 How is the first equality justified? Re-expressing the *terms* is allowable, but is re-expressing the limit permissible?
- 3 Is this the limit of the terms or the limit of the sequence?

Admittedly the last concern is arguably picky, or at best an issue of notation. But NOT the first two!

Why unnecessarily let students use objects whose existence is yet uncertain when they are freshmen, but forbid it later?

A (Sketch of a) Proper Way

- 1 LL1 (Sums of Convergent Sequences); LL4 (Quotients of Convergent Sequences, appropriately restricted)
- 2 Theorem ESEL (Equal Sequences \Rightarrow Equal Limits)

$$\left(\frac{1}{n}\right) \rightarrow 0$$

Proven in class

$$(1) \rightarrow 1$$

(A)

Theorem Const.

$$\left(\frac{1}{n} + 1\right) \rightarrow 0 + 1 = 1 \quad (B)$$

LL1 and (A)

$$\left(\frac{1}{\frac{1}{n} + 1}\right) \rightarrow \frac{1}{0 + 1} = 1 \quad (C)$$

LL4 and (B)

$$\left(\frac{n}{n+1}\right) \rightarrow 1$$

(C) and Theorem ESEL

A Typical Example

In S & S, when students are asked to determine if $\left(\frac{\sin n}{n^{\frac{3}{2}}}\right)$ converges, they might note first that

$$-1 \leq \sin n \leq 1 \quad (\text{known fact about the range of sine})$$

Now since $n > 0$, we have $n^{\frac{3}{2}} > 0$ so that

$$\frac{-1}{n^{\frac{3}{2}}} \leq \frac{\sin n}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}} \quad (1)$$

follows. Since $\left(\frac{-1}{n^{\frac{3}{2}}}\right)$ and $\left(\frac{1}{n^{\frac{3}{2}}}\right)$ are both p -sequences with $p = 3/2 > 0$, both converge to 0. Thus by Eq. (1) and the Squeeze Theorem, we conclude that $\left(\frac{\sin n}{n^{\frac{3}{2}}}\right)$ converges to 0 as well.

The Natural Role of S & S in a Small University Without Engineering Programs

The main point of this talk is the following:

Sequences and Series is a great setting in which to foster students' proofs skills before the problems get more elaborate

In any course in which one expects rigor, students need to know *exactly what they're allowed to use as supporting evidence.*

To Make S & S a Proofs Course, What is Sufficient?

- 1 First Principle - treat all theorems and convergence tests as *axioms*
 - We're not proving broadly applicable theorems
- 2 Second Principle - Explicitly indicate what are to be considered *known facts*
 - Facts about domain, range, and continuity for trigs, arctrigs, exponentials, logarithms, inequalities, etc.
- 3 Third Principle - Ensure that definitions, axioms, etc., are precise and sufficiently broad - no loopholes!

To Make S & S Logically Self-Contained, What is Sufficient?

A variety of simple but loophole-closing theorems (again, treated as axioms) - Several are listed here but all are available on my website on the last slide:

- 1 The Continuous Function Theorem - for functions continuous at $x_0 \in \mathbb{R}$, this permits us to pass the limit through the function to the argument; makes showing $\sin(1/n) \rightarrow 0$ rigorous but trivial
- 2 Precise definitions of *boundedness* and *unboundedness* - these permit one to treat divergent sequences more carefully through the theorems shown on the next slide

Divergence Deserves Respect, Too

To show that a sequence diverges:

- The Push Theorem - Analogous to the Squeeze Theorem, but for divergent sequences
- The Subsequence Theorem - All subsequences of convergent sequences converge to the common limit; the converse permits determination of divergence for a broad range of sequences

The Monotonic Sequence Theorem

- Give examples of propositional functions on \mathbb{N}
- Give examples of mathematical induction, with the domino analogy
- Note that each of the two hypotheses of the MST are universally quantified on (at least a tail of) \mathbb{N}
- My particular approach is to ask that they use the following algorithm for each demonstration of hypothesis satisfaction:
 - 1 Identify the relevant propositional function
 - 2 Base case
 - 3 Induction hypothesis
 - 4 Induction step
 - 5 Conclusion, carefully expressed

MST Continued

These problems are a gold mine!

What does it even mean to “take the limit of both sides” of

$$a_{n+1} = 3 - \frac{1}{a_n}?$$

We’ll need ESEL, and to cite the Subsequence Theorem as justification for saying $(a_n) \rightarrow \ell \Rightarrow (a_{n+1}) \rightarrow \ell$

Also, in the spirit of rigor, we need to *carefully* rule out the extraneous root of the quadratic equation $\ell = 3 - \frac{1}{\ell}$

This permits us to introduce the important result that the limit of a convergent sequence can’t escape the closure of the sequence’s range

Series

Most of the main convergence tests for series rely of the convergence of some *sequence* derived from the series at hand

The Test for Divergence, the Limit Comparison Test, the Ratio Test, the Alternating Series Test and finding the sum of convergent series with telescoping partial sums all fall into this category - *this is why I spend so much time on the sequence portion of the course*

We also note that the Ratio Test with nonzero, finite limit indicates a series which “acts” like an equivalent geometric series as n grows without bound

Rigor in the Power Series Sections of the Course

(Multiplication of Power Series by a Monomial - MPSM): Let $\sum c_n x^n$ be a PSR, valid on $(-R, R)$, for a function f , and suppose that the lowest-degree monomial in said PSR is x^m for some $m \in \mathbb{N} \cup \{0\}$. Let $k \in \mathbb{Z}$ (that is, k is an integer, perhaps negative) with $m + k \geq 0$. Then a PSR for $x^k f(x)$, $k \in \mathbb{N}$ is $\sum c_n x^{n+k}$, valid on $(-R, R)$.

This theorem is useful for justifying the obvious manner of constructing series to represent functions of the form $\frac{x^3}{1-x}$ or $x^2 \sin x$, etc.

Rigor in the Power Series Sections of the Course

(Power Series Substitution - PSS): Let a be a (typically nonzero) constant. If $\sum c_n x^n$ is a PSR for $f(x)$, valid on $(-R, R)$, then a PSR for $f(ax^k)$, $k \in \mathbb{N}$ is $\sum c_n a^n x^{nk}$, valid on $\left(-\sqrt[k]{\frac{R}{|a|}}, \sqrt[k]{\frac{R}{|a|}}\right)$,

This theorem is useful for producing series to represent functions of the form $\frac{1}{1 \pm x^k}$, or $\sin x^3$, etc., or in combination with MPSM, $\frac{x^3}{1 \pm x^k}$, $x^2 \cos \pi x^4$, etc.

Approximate Duration Requirements of S & S as Described Herein

- 1 Spend one or two days on logic: propositions, conditional propositions, converse, contrapositive; just the basics
 - Propositional functions, particularly if one plans to cover the Monotonic Sequence Theorem
- 2 Three weeks on sequences (four if covering the MST)
- 3 Three weeks on numerical series
 - Since Taylor series is such a rich topic, one may choose to spend only two weeks on numerical series, foregoing several convergence tests in favor of Taylor series applications
- 4 Three weeks on Taylor series (or four; see above)

Is it Worth It?

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- Yes.

Contact Information

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(My S & S class notes in pdf and \LaTeX form are available there)