Math 344: Binary Operations

In this context, we refer to the operation, *not the set*, as being defined and well-defined. Specifically, the usual phrasing is "(operation) is defined/well-defined on (set)." For closure, there are two standard options "(operation) closed on (set)" or "(set) is closed under (operation)."[1]

1. In each of the following, a set and an alleged operation on that set are given.[2] In each case, decide if the alleged operation really is an operation (defined, well-defined, and closed) or not. Then carefully prove your assertion by either showing that all three properties do hold or by showing that one of the properties fails. Watch the quantifiers.

   (a) $\mathbb{Q}$ (the set of rational numbers); $\diamond$ where $a \diamond b = \sqrt{|ab|}$ for every $a, b \in \mathbb{Q}$.

   (b) $\mathbb{Z}^*$ (the set of nonzero integers); $\div$ (ordinary division)

   (c) $\mathbb{Q}^*$ (the set of nonzero rational numbers); $\div$ (ordinary division)

   (d) $\mathbb{R}$ (the set of real numbers); $\otimes$ where $r \otimes s = r \ln(s)$ for every $r, s \in \mathbb{R}$.

   (e) $\mathbb{C}$; $\ominus$ where for every $u, v \in \mathbb{C}$,
   $u \ominus v$ equals one of the solutions of the equation $x^2 - 2(u+v)x + 4uv = 0$.

   (f) $\mathbb{R}^*$ (the set of nonzero real numbers); $\odot$ where $y \odot z = \dfrac{yz}{2}$ for every $r, s \in \mathbb{R}^*$.

2. Give two sets, at least one of which is finite, (not listed above) for which division IS an operation on the set.

3. Give two sets (not listed above) for which addition is NOT an operation on the set.

4. Give an original example of a set with an alleged operation that is defined and closed, but not well-defined.

[1] Some of this worksheet comes from Mike Ward

[2] Some of these "operations" are from *A Book of Abstract Algebra* by Charles C. Pinter.