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Ask Marilyn™

BY MARILYN VOS SAVANT

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
—Craig F. Whitaker, Columbia, Md.

Yes; you should switch. The first door has a one-third chance of winning, but the second door has a two-thirds chance. Here's a good way to visualize what happened. Suppose there are a *million* doors, and you pick door No. 1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door #777,777. You'd switch to that door pretty fast, wouldn't you?

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I'll come straight to the point. In the following question and answer, you blew it!

"You're on a game show and given a choice of three doors. Behind one is a car; behind the others are goats. You pick Door No. 1, and the host, who knows what's behind them, opens No. 3, which has a goat. He then asks if you want to pick No. 2. Should you switch?"

You answered, "Yes. The first door has a 1/3 chance of winning, but the second has a 2/3 chance."

Let me explain: If one door is shown to be a loser, that information changes the probability of either remaining choice to 1/2. As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and, in the future, being more careful.
—Robert Sachs, Ph.D., George Mason University, Fairfax, Va.

You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain: After the host reveals a goat, you now have a one-in-two chance of being correct. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!
—Scott Smith, Ph.D., University of Florida

Your answer to the question is in error. But if it is any consolation, many of my colleagues have also been stumped by this problem.
—Barry Pasternack, Ph.D., California Faculty Association

Good heavens! With so much learned opposition, I'll bet this one is going to keep math classes all over the country busy on Monday.

My original answer is correct. But first, let me explain why your answer is wrong. The winning odds of 1/3 on the first choice can't go up to 1/2 just because the host opens a losing door. To illustrate this, let's say we play a shell game. You look away, and I put a pea under one of three shells. Then I ask you to put your finger on a shell. The odds that your choice contains a pea are 1/3, agreed? Then I simply lift up an empty shell from the remaining two. As I can (and will) do this regardless of what you've chosen, we've learned nothing to allow us to revise the odds on the shell under your finger.

The benefits of switching are readily proved by playing through the six games that exhaust all the possibilities. For the first three games, you choose No. 1 and switch each time; for the second three games, you choose No. 1 and "stay" each time, and the host always opens a loser. Here are the results (each row is a game):

DOOR 1	DOOR 2	DOOR 3
AUTO	GOAT	GOAT
Switch and you lose.		
GOAT	AUTO	GOAT
Switch and you win.		
GOAT	GOAT	AUTO
Switch and you win.		
AUTO	GOAT	GOAT
Stay and you win.		
GOAT	AUTO	GOAT
Stay and you lose.		
GOAT	GOAT	AUTO
Stay and you lose.		

When you switch, you win two out of three times and lose one time in three; but when you don't switch, you only win one in three times and lose two in three. Try it yourself.

Alternatively, you actually can play the game with another person acting as the host with three playing cards—two jokers for the goats and an ace for the auto. However, doing this a few hundred times to get statistically valid results can get a little tedious, so perhaps you can assign it for extra credit—or for punishment! (*That'll* get their goats!)

If you have a question for Marilyn vos Savant, who is listed in the "Guinness Book of World Records Hall of Fame" for "Highest IQ," send it to: Ask Marilyn, PARADE, 750 Third Ave., New York, N.Y. 10017. Because of volume of mail, personal replies are not possible.

Ask Marilyn™

BY MARILYN VOS SAVANT

You are in error—and you have ignored good counsel—but Albert Einstein earned a dearer place in the hearts of the people after he admitted his errors.
—Frank Rose, Ph.D., University of Michigan

I have been a faithful reader of your column and have not, until now, had any reason to doubt you. However, in this matter, in which I do have expertise, your answer is clearly at odds with the truth.
—James Rauff, Ph.D, Millikin University

May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?
—Charles Reid, Ph.D., University of Florida

Your logic is in error, and I am sure you will receive many letters on this topic from high school and college students. Perhaps you should keep a few addresses for help with future columns.
—W. Robert Smith, Ph.D. Georgia State University

You are utterly incorrect about the game-show question, and I hope this controversy will call some public attention to the serious national crisis in mathematical education. If you can admit your error, you will have contributed constructively toward the solution of a deplorable situation. How many irate mathematicians are needed to get you to change your mind?
—E. Ray Bobo, Ph.D., Georgetown University

I am in shock that after being corrected by at least three mathematicians, you still do not see your mistake.
—Kent Ford, Dickinson State University

Maybe women look at math problems differently than men.
—Don Edwards, Sunriver, Ore.

You are the goat!
—Glenn Calkins Western State College

You're wrong, but look at the positive side. If all those Ph.D.s were wrong, the country would be in very serious trouble.
—Everett Harman, Ph.D., U.S. Army Research Institute

Gasp! If this controversy continues, even the *postman* won't be able to fit into the mailroom. I'm receiving thousands of letters, nearly all insisting that I'm wrong, including one from the deputy director of the Center for Defense Information and another from a research mathematical statistician from the National Institutes of Health! Of the letters from the general public, 92% are against my answer; and of the letters from universities, 65% are against my answer. Overall, nine out of 10 readers completely disagree with my reply.

But math answers aren't determined by votes. For those readers new to all this, here's the original question and answer in full, to which the first readers responded:

"Suppose you're on a game show, and you're given a choice of three doors. Behind one door is a car; behind the others, goats. You pick a door—say, No.1—and the host, who knows what's behind the doors, opens another door—say, No. 3—which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to switch your choice?"

I answered, "Yes, you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance. Here's a good way to visualize what happened. Suppose there are a *million* doors, and you pick door No. 1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door No. 777,777. You'd switch to that door pretty fast, wouldn't you?"

So many readers wrote to say they thought there was *no* advantage to switching (and that the chances became equal) that we published a second explanatory column, affirming the correctness of the original reply and using a shell game and a probability grid as illustrations.

Now we're receiving far *more* mail, and even newspaper columnists are joining in the fray. The day after the second column appeared, lights started flashing here at the magazine. Telephone calls poured into the switchboard, fax machines churned out copy, and the mailroom began to sink under its own weight. Incredulous at the response, we read wild accusations of intellectual irresponsibility and, as the days went by, we were even more incredulous to read embarrassed retractions from some of those same people!

The reaction is understandable. When reality clashes so violently with intuition, people are shaken.

But understanding is strength, so let's look at it again, remembering that the original answer defines certain conditions—the most significant of which is that *the host will always open a losing door on purpose.* (There's no way he can always open a losing door by chance!) Anything else is a different question.

The original answer is still correct, and the key to it lies in the question: *Should you switch?* Suppose we pause at that point, and a UFO settles down onto the stage. A little green woman emerges, and the host asks her to point to one of the two unopened doors. The chances that *she'll* randomly choose the one with the prize are 1/2. But that's because she lacks the advantage the *original* contestant had—the help of the host. (Try to forget any particular television show.)

When you first choose door No. 1 from among the three, there's a 1/3 chance that the prize is behind that one and a 2/3 chance that it's behind one of the others. *But then the host steps in and gives you a clue.* If the prize is behind No. 2, the host shows you No. 3; and if the prize is behind No. 3, the host shows you No. 2. So when you switch, you win if the prize is behind No. 2 *or* No. 3. *YOU WIN EITHER WAY!* But if you *don't* switch, you win only if the prize is behind door No. 1.

And as this problem is of such intense interest, I'm willing to put my thinking to the test with a nationwide experiment. This is a call to math classes all across the country. Set up a probability trial exactly as outlined below and send me a chart of all the games, along with a cover letter repeating just how you did it, so we can make sure the methods are consistent.

One student plays the contestant, another plays the host. Label three paper cups No. 1, No. 2 and No. 3. While the contestant looks away, the host randomly hides a penny under a cup by throwing a die until a 1, 2 or 3 comes up. Next, the contestant randomly points to a cup by throwing a die the same way. Then the host purposely lifts up a losing cup from the two unchosen. Last, the contestant "stays" and lifts up his original cup to see if it covers the penny. Play "not switching" 200 times and keep track of how often the contestant wins.

Then test the other strategy. Play the game the same way until the last instruction, at which point the contestant instead "switches" and lifts up the cup *not* chosen by anyone to see if it covers the p[enny]. Play "switching" 200 times also.

And here's one last letter:

Dear Marilyn:
You are indeed correct. My colleagues at work had a ball with this problem, and I dare say that most of them—including me at first—thought you were wrong!
—Seth Kalson, Ph.D., Massachusetts Institute of Technology

Thanks, MIT. I needed that!

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Ask Marilyn™

BY MARILYN VOS SAVANT

In a recent column, you called on math classes around the country to perform an experiment that would confirm your response to a game-show problem. ["Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to switch your choice?"]

You answered, "Yes, you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance. Here's a good way to visualize what happened: Suppose there are a *million* doors, and you pick door No. 1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door No. 777,777. You'd switch to that door pretty fast, wouldn't you?"

My eighth-grade classes tried it [switching and not switching, 200 times each, using three cups and a coin]. I don't really understand how to set up an equation for your theory, but it definitely does work! You'll have to help rewrite the chapters on probability.

—Pat Gross, Ascension School, Chesterfield, Mo.

Our class, with unbridled enthusiasm, is proud to announce that our data support your position. Thank you so much for your faith in America's educators to solve this.

—Jackie Charles, Henry Grady Elementary, Tampa, Fla.

My class had a great time watching your theory come to life. I wish you could have been here to witness it. Their joy is what makes teaching worthwhile.

—Pat Pascoli, Park View School, Wheeling, W.Va

Seven groups worked on the probability problem. The numbers were impressive, and the students were astounded.

—R. Burrichter, Webster Elementary School, St. Paul, Minn.

The best part was seeing the looks on the students' faces as their numbers were tallied. The results were thrilling!

—Patricia Robinson, Ridge High School, Basking Ridge, N.J.

You could hear the kids gasp, one at a time, "Oh, my gosh! She was right!"

—Jane Griffith, Magnolia School, Oakdale, Calif.

I must admit I doubted you until my fifth-grade math class proved you right. All I can say is WOW!

—John Witt, Westside Elementary, River Falls, Wis.

My classes enjoyed this and look forward to the next project you give America's students. This is the stuff of real science.

—Jerome Yeutter, Hebron Public Schools, Hebron, Neb.

Thanks for that fun math problem. I really enjoyed it. It got me out of fractions for two days! Have any more?

—Andrew Malinoski, Mabelle Avery School, Somers, Conn.

I did your experiment on probability as part of a science-fair project, and after extensive interviews with the judges, I was awarded first place.

—Adrienne Shelton, Holy Spirit School, Annandale, Va.

I also thought you were wrong, so I did your experiment, and you were exactly correct. (I used three cups to represent the three doors, but instead of a penny, I chose an aspirin tablet because I thought I might need to take it after my experiment.)

—William Hunt, M.D., West Palm Beach, Fla.

I put my solution of the problem on the bulletin board in the Physics Department office here, following it with a declaration that you were right. All morning I took a lot of criticism and abuse from my colleagues, but by late in the afternoon most of them came around. I even won a free dinner from one overconfident professor.

—Eugene Mosca, Ph.D., U.S. Naval Academy, Annapolis, Md.

After considerable discussion and vacillation here at the Los Alamos National Laboratory, two of my colleagues independently programmed the problem, and in one million trials, switching paid off 66.7% of the time. The total running time on the computer was less than one second.

—G.P. DeVault, Ph.D, Los Alamos National Laboratory, Los Alamos, N.M.

Now fess up. Did you really figure all this out, or did you get help from a mathematician?

—Lawrence Bryan, San Jose, Calif.

Wow! What a response we received! It's still coming in, but so many of you are so anxious to hear the results that we'll stop tallying for a moment and take stock of the situation so far. We've received thousands of letters, and of the people who performed the experiment by hand as described, the results were close to unanimous: You win twice as often when you change doors. Nearly 100% of those readers now believe it pays to switch. But many people tried performing similar experiments on computers, fearlessly programming them in hundreds of different ways. Not surprisingly, they fared a little less well. Even so, about 97% of them now believe it pays to switch.

And plenty of people who *didn't* perform the experiment wrote too. Of the general public, about 56% now believe you should switch, compared with only 8% before. From academic institutions, about 71% now believe you should switch, compared with only 35% before. (Many of them commented that it altered their thinking dramatically, especially about the state of mathematical education in this country.) And a very small percentage of readers feel convinced that the furor is resulting from people not realizing that the host is opening a losing door on purpose. (But they haven't read my mail! The great majority of people understand the conditions perfectly.)

And so we've made progress! Half of the readers whose letters were published in the previous columns have written to say they've changed their minds. But, of course...

Dear Marilyn:
I still think you're wrong. There is such a thing as female logic.

—Don Edwards, Sunriver, Ore.

Oh, hush now.

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The *Car-and-Goats* FIASCO

A number of mathematicians were thrown into a tizzy by the following problem, which appeared last fall in Marilyn vos Savant's column, "Ask Marilyn," in *Parade* (a Sunday newspaper supplement):

> *One of three doors hides a car (all three equally likely) and the other two hide goats. You choose Door 1. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and asks whether you wish to switch your choice. Say he opens Door 3; should you switch to Door 2?*

Marilyn said yes, arguing that the probability for Door 2 is now $2/3$. This led to protests, another column, and then a deluge of additional protests. Altogether she heard from "thousands" of people. Nine-tenths of them insisted that with Door 3 now eliminated, Doors 1 and 2 were still equally likely.

Of the respondents from the general public, 92% disagreed with her, while of the responses from universities, 65% disagreed. It follows that $7\frac{1}{2}$% of the responses came from universities. Therefore 5% were naysayers from universities, and one may suppose that just about all of this group were professional mathematicians.

The mathematicians were a disgrace to the profession. The puzzle requires more patience than readers predisposed toward the rough-and-ready answer seemed willing to devote. Many writers appear to have rushed to reply without first letting their ideas jell. I saw no evidence of the patient attitude of a *teacher*. Instead, the writers quoted were arrogant and condescending—"Get yourself a standard book on probability," "You're the goat," "We're having enough trouble with mathematical illiteracy as it is"—and then proudly identified themselves as PhDs or faculty members. One of them asserted that if the PhD critics were wrong then the country would be in serious trouble.

The country is in serious trouble.

These writers are the products of our graduate schools. Is that where they acquire these attitudes? In class? From professors they see rushing into print?

Interestingly enough, no one seemed to notice that Marilyn had introduced not one game but two. Game I, stated above, is the one the writers are upset about.

As for Game II, Marilyn defined it implicitly by just solving it: If the car is actually at Door 1 (probability $1/3$), then when you switch you lose; but if it is at Door 2 *or* Door 3 (probability $2/3$) then the host's revelation of a goat shows you how to switch and win. Therefore the chance you win by switching is $2/3$. Elegant. But in this argument, we are still considering the possibility that the car is behind Door 3; so the host cannot have already opened that door (much less to reveal a goat). In this game, you have to announce *before a door has been opened* whether you intend to switch.

Game I is a different matter. Here the probability, P, that switching wins is a *conditional* probability: that it wins *given that the host has opened Door 3*. It is easy to see that $P \geq 1/2$. (So the critics are quite wrong.) The host has opened Door 3. It was *certain* he would do that if the car is at Door 2, but less than certain (except in an extreme case) if it is at Door 1. This gives the edge to Door 2. (This reasoning depends on the fact that the *a priori* probabilities for the two doors are equal.)

In fact, *P can be any number between $1/2$ and 1.* The nub of the matter is what happens when the car is actually at Door 1, so tha the host has a choice of doors to open. The value of P depends on how he makes that choice—on the *probability*, q, that he will open Door 3.

In the extreme case, $q = 0$, the host opens Door 3 only when the car is at Door 2, and $P = 1$.

When $q = 1/2$ we get $P = 2/3$. For when the car is at Door 1 the host opens Door 3 one time in two; but if it is at Door 2 he opens Door 3 two times in two. So when he actually does open Door 3, the car is at Door 2 two times out of three.

Similarly, $P = 1/(1+q) = n/(n+m)$ for any rational $q = m/n$. (By Bayes's rule, the first equation holds for all real q.) Of course, $0 \leq q \leq 1$ implies $1 \geq P \geq 1/2$.

Over a long series of games, where the host opens Door 3 or Door 2 according to his strategy and you switch every time, your win probability is $2/3$. This is true regardless of how he chooses or mixes his strategies. Say he sticks with the strategy of the first example. Then he opens Door 3, giving you the sure shot, only $1/3$ of the time; the remaining $2/3$ of the time, when he opens Door 2, your win probability is $1/2$. Your net chances are $1/3$ in each case, for a total of $2/3$. This is no surprise: you are now playing Game II. ■

LEONARD GILLMAN

Leonard Gillman is Professor Emeritus at the University of Texas at Austin. His many services to the Association include terms as MAA Treasurer (1973–1985) and MAA President (1987–1988). Readers interested in the car-and-goats problem will discover a fuller account of it mathematics in the January 1992 issue of the MAA's American Mathematical Monthly.