Winning Lines in Tic-Tac-Toe

1. The standard Tic-Tac-Toe is played on a 3 x 3 board, where there are _____ vertical winning lines, ______ horizontal winning lines, ______ diagonal winning lines. This is a grand total of _______ winning lines in the standard 3x3 Tic-Tac-Toe.

2. In 4 x 4 Tic-Tac-Toe, the winning lines are again 4-in-a-row horizontally, vertically or diagonally. How many winning lines are there in 4 x 4 Tic-Tac-Toe? ___________________

3. How many winning lines are there in 5x5 Tic-Tac-Toe? _________________

4. How many winning lines are there in n x n Tic-Tac-Toe? _________________

Pairing Strategies in n x n Tic-Tac-Toe:

Below is what is called a Pairing Strategy Draw for Player 2 in 6 x 6 Tic-Tac-Toe:

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Can you create a Pairing Strategy Draw for Player 2 in 5 x 5 Tic-Tac-Toe?
Can you create a Pairing Strategy Draw for Player 2 in 4 x 4 Tic-Tac-Toe?

Can you create a Pairing Strategy Draw for Player 2 in 3 x 3 Tic-Tac-Toe?

**Q** What do you need to happen to HOPE that a Pairing Strategy Draw exists for Player 2?

**Notation:** (To hopefully make our life easier…..) We will consider Tic-Tac-Toe to be played by two players who select points from the plane. In $n \times n$ Tic-Tac-Toe, the points the players select are in the set $\{(1,1), (1,2), \ldots, (1,n), (2,1), (2,2), \ldots, (2,n), \ldots, (n,n)\}$ This type of notation may help us when we play Tic-Tac-Toe in higher dimensions.

**Note:**
- When we think of a horizontal winning line in Tic-Tac-Toe, we can think of this as a series of points, where the **y-coordinate is fixed** and the **x-coordinate varies**.

- Similarly, a vertical winning line has the **x-coordinate fixed** and the **y-coordinate varying**.

- A diagonal winning line has both the **x-coordinate and the y-coordinate varying**.
You may know about three-dimensional Tic-Tac-Toe, where the winning lines either have one, two or three coordinates varying.

**Notation:** In the interest of my sanity, we will say \( n^3 \text{ TTT} \) is \( n \times n \times n \) Tic-Tac-Toe. This game will be played on a cube where each side is of length \( n \).

5. How many winning lines are there in \( 2^3 \text{ TTT} \)?

6. How many winning lines are there in \( 3^3 \text{ TTT} \)?

7. How many winning lines are there in \( 4^3 \text{ TTT} \)?

8. Find a formula for the number of winning lines in \( n^3 \text{ TTT} \).

**Tricks**
Here are two ways for determining the number of winning lines in \( n^d \text{ TTT} \) (d-dimensional Tic-Tac-Toe where each winning line is of length \( n \)). For this game, \( n \) and \( d \) are any positive integers and the game is played on an \( n^d \) hypercube.

a. Consider each winning line as having d-coordinates (since this is d-dimensional Tic-Tac-Toe) where each coordinate is element of the set \( \{1, 2, \ldots, n, \uparrow, \downarrow\} \).
   \( \uparrow \) means the coordinate is varying **up**.
   \( \downarrow \) means the coordinate is varying **down**.

   How many winning lines are there if at least one coordinate is varying?
b. In 3x3 Tic-Tac-Toe, draw a one square border around the board. Draw in your complete winning lines for the 3x3 board and extend them into your 5 x 5 rectangle. Notice that each winning line hits exactly two points in the shaded border.

How many squares are there in the shaded region? ______________
How many winning lines were there in 3x3 Tic-Tac-Toe? __________

If you draw in a one square border around the 7x7 Tic-Tac-Toe board and how many squares would there be in the border? ______________
How many winning lines are there in 7x7 Tic-Tac-Toe? ____________

If you draw in a one square border around the n x n Tic-Tac-Toe board and how many squares would there be in the border? ______________
How many winning lines are there in n x n Tic-Tac-Toe? ____________

**Trying the same trick for n^d board.**

How many winning lines are there in the n^3 board…..

using **Trick a**? ____________________________

using **Trick b**? ____________________________

**The number of directions** in n^d TTT:

In n x n Tic-Tac-Toe, there are four different directions: ↑, ↓, \, /. 

How many directions are there in n^3 TTT? ______________
How many directions are there in n^4 TTT? ______________
How many directions are there in n^d TTT? ______________
Tic-Tac-Toe on the Torus

Pretend that the Tic-Tac-Toe is wrapped around a sphere, so that there is no top/bottom of the board or left/right of the board. All squares are effectively equivalent.

Winning Lines in 3 x 3 Tic-Tac-Toe on a torus:
Besides the 8 usual lines, there are the following additional lines.

Let’s play the game: We know that Player 2 can force a draw in standard 3x3 Tic-Tac-Toe.

Q: Can Player 2 force a draw in 3 x 3 Tic-Tac-Toe on a torus? ___________

Play a few games with your classmates. See if you can figure it out. Then analyze the game move by move.

Winning Lines in Tic-Tac-Toe on a torus

How many winning lines are there in 4 x 4 Tic-Tac-Toe on a torus? ___________

How many winning lines are there in 5 x 5 Tic-Tac-Toe on a torus? ___________

How many winning lines are there in \( n \times n \) Tic-Tac-Toe on a torus? ___________

How many winning lines are there in \( n^3 \) Tic-Tac-Toe on a torus? ________________