The Fourier Transform, $L^2(\mathbb{R})$, and the Riemann-Lebesgue Lemma

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The Fourier Transform on $L^1(\mathbb{R})$

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} \, dx$$  \hspace{1cm} (1)$$

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) := \int_{\mathbb{R}} g(\omega) e^{2\pi i \omega x} \, d\omega$$  \hspace{1cm} (2)$$

These are valid wherever the integrals are defined, for example if $f \in L^1(\mathbb{R})$
$C_0(\mathbb{R})$ and $S$

**Definition**

$C_0(\mathbb{R})$ is the set of continuous functions with a horizontal asymptote at $f = 0$ as $|x| \to \infty$.

**Definition**

The Schwartz Space $S$ is defined as the space of $C^\infty$ functions on $\mathbb{R}$ which, along with all of their derivatives, decay faster than any rational function.

**Theorems**

$S \xrightarrow{d} L^1(\mathbb{R})$; $C_0(\mathbb{R}) \subset L^\infty(\mathbb{R})$; $\mathcal{F}: S \to S$ is an isomorphism.
The Riemann-Lebesgue Lemma

**Theorem**

\[ f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R}) \]

The integral can be expected to be vanishingly small as the frequency increases without bound.
Proof of RLL (after Nachtergaele)

Proof

By density, choose \((g_n)\) from \(S\) converging to \(f\) in \(L^1(\mathbb{R})\); then \((\hat{g}_n)\) is (uniformly) Cauchy:

\[
|\hat{g}_n(\omega) - \hat{g}_m(\omega)| = \left| \int_{\mathbb{R}} (g_n(x) - g_m(x)) e^{-2\pi i \omega x} \, dx \right| \\
\leq \int_{\mathbb{R}} |g_n(x) - g_m(x)| \, dx \\
= \|g_n - g_m\|_1
\]

Since \(S \subset C_0(\mathbb{R})\) which is complete under the sup norm, there exists a function \(h \in C_0(\mathbb{R})\) to which \((\hat{g}_n)\) converges uniformly.
Proof (Cont’d)

Finally, \( h = \hat{f} \) since \( \forall \omega \in \mathbb{R} \),

\[
\left| h(\omega) - \hat{f}(\omega) \right| = \lim_{n \to \infty} \left| \hat{g}_n(\omega) - \hat{f}(\omega) \right|
\]

\[
= \lim_{n \to \infty} \left| \int_{\mathbb{R}} (g_n(x) - f(x))e^{-2\pi i \omega x} \, dx \right|
\]

\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}} |g_n(x) - f(x)| \, dx = 0
\]
The Fourier Transform on $L^2(\mathbb{R})$

For $f \in L^2(\mathbb{R})$, we cannot in general define $\hat{f}$ by Equation (1).

**Theorem**

*(Plancheral’s Theorem)* $f \in L^2(\mathbb{R}) \Rightarrow \|\hat{f}\|_2 = \|f\|_2$.

Now $(L^1 \cap L^2)(\mathbb{R}) \xrightarrow{d} L^2(\mathbb{R})$, so we can choose a sequence $(f_n)$ in $(L^1 \cap L^2)(\mathbb{R})$ which converges to $f$ in the 2-norm.
The Fourier Transform on $L^2(\mathbb{R})$

By Plancheral’s Theorem, $\|f_n - f_m\|_2 = \|\hat{f}_n - \hat{f}_m\|_2$, so $(\hat{f}_n)$ is Cauchy in $L^2(\mathbb{R})$, hence converges.

We now define $\hat{f} := \lim_{n \to \infty} \hat{f}_n$.

The inverse Fourier transform $\mathcal{F}^{-1}$ is then defined as the Hilbert space adjoint $\mathcal{F}^*$ of $\mathcal{F}$ (but not necessarily as a pointwise formula).
Fourier Inversion

First note that (by RLL) \( f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C_0(\mathbb{R}) \)

**Theorem**

If \( f, \hat{f} \in L^1(\mathbb{R}) \), then \( f(x) = \int_{\mathbb{R}} \hat{f}(\omega) e^{2\pi i \omega x} \, d\omega \)

So if \( f, \hat{f} \in L^1(\mathbb{R}) \), then \( f, \hat{f} \in C_0(\mathbb{R}) \)
Useful Sufficient Conditions for \((\mathcal{F}^{-1} \circ \mathcal{F})(f) = f\)

In this case we have \(f, \hat{f} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})\), and splitting \(\mathbb{R}\) into complementary sets for which \(f\) (resp. \(\hat{f}\)) \(\geq 1\) or \(f < 1\) yields that

\[
f, \hat{f} \in L^1(\mathbb{R}) \Rightarrow f, \hat{f} \in (L^2 \cap C_0)(\mathbb{R}) \Rightarrow f, \hat{f} \in L^1(\mathbb{R})
\]

So on a significant subset of \(L^2(\mathbb{R})\), \(\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{I}\)

This set includes \(e^{-x^2}\), \(\text{sinc}^2(x)\), which are formulaic, and many others which are not
The Chernoff-Fourier Convergence Theorem

(From Amer. Math Monthly (1980), 399-400)

**Theorem**

Let $f$ be absolutely integrable on an interval $I$ and suppose $f$ is Lipschitz on $I$ with constant $A$. Then the (asymmetric) partial sums

$$S_{m,n}(x_0) := \sum_{k=-m}^{n} \hat{f}(k)e^{\frac{2\pi ikx_0}{\ell(I)}}$$

converge to $f(x_0)$ as $m, n \rightarrow +\infty$. 
Proof of Pointwise Fourier Series Convergence

Proof

WLOG we can suppose that $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, that $x_0 = 0$, and that $f(0) = 0$. Consider the auxiliary function $g(x) := \frac{f(x)}{e^{2\pi ix} - 1}$ and note

$$\left| \frac{f(x)}{e^{2\pi ix} - 1} \right| = \left| \frac{f(x)}{x} \frac{x}{e^{2\pi ix} - 1} \right| \leq A \cdot \left| \frac{x}{e^{2\pi ix} - 1} \right|$$

Also, $\left| \frac{e^{2\pi ix} - 1}{x} \right| \geq 4 \Rightarrow \left| \frac{x}{e^{2\pi ix} - 1} \right| \leq \frac{1}{4}$
Proof of Pointwise Fourier Series Convergence

Proof (Cont’d)

Now note that \( \hat{f}(k) = \mathcal{F}(g \cdot (e^{2\pi i x} - 1))(k) = \hat{g}(k - 1) - \hat{g}(k) \) which expresses \( \hat{f} \), so

\[
S_{m,n}(0) = \sum_{k=-m}^{n} \hat{f}(k)e^{2\pi ik(0)} = \hat{g}(-m - 1) - \hat{g}(n) \tag{3}
\]

which converges to \( 0 - 0 = 0 = f(0) \) by the Riemann-Lebesgue Lemma applied to \( g \).
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