1. Find integers $a_1, a_2, a_3, a_4,$ and $a_5$, such that every integer $x$ satisfies at least one of the following congruences:

\[
\begin{align*}
    x &\equiv a_1 \pmod{2} \text{ and/or} \\
    x &\equiv a_2 \pmod{3} \text{ and/or} \\
    x &\equiv a_3 \pmod{4} \text{ and/or} \\
    x &\equiv a_4 \pmod{6} \text{ and/or} \\
    x &\equiv a_5 \pmod{12}
\end{align*}
\]

Prove your results!

2. Show that the square of every odd integer is of the form $8k + 1$.

3. Let $p$ be a prime $p > 5$. What are the possible values $0 \leq a < 30$ so that $p \equiv a \pmod{30}$. Prove why these are the correct values.

4. Let $p$ be prime, then prove that $a > 0$ is its own inverse modulo $p$ if and only if $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$.

5. If $\gcd(a, n) = 1$, show that every integer is a multiple of $a \pmod{n}$.

6. Let $x$ be a positive integer. Show that $x$ is divisible by 3 if and only if the sum of the digits of $x$ is divisible by 3.

7. In make up problem 6, you (may or may not have) proved that a number $n$ is divisible by 3 if and only if the sum of its digits is divisible by 3. Find and prove a similar rule for determining if a number $n$ is divisible by 11. Your statement should say something like “$n$ is divisible by 11 if and only if “...” where in place of “...” should be something involving the digits of 11.

8. Explain how you could use the Chinese Remainder Theorem when finding the solution(s) to the congruence $x^2 + 6x - 31 \equiv 0 \pmod{72}$. Then use this method to find a solution. Show work.

9. Use the Chinese Remainder Theorem to find the inverse of 68 modulo 77. Show all work. You must use the Chinese Remainder Theorem in your solution or you will get no credit.

10. Show that $a^{12} - 1$ is divisible by 35 whenever $\gcd(a, 35) = 1$. (Hint: You will use Fermat’s little theorem twice.)