New Practices in Teaching Mathematical Definitions

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INTRODUCTION

The National Council of Teachers of Mathematics Communication Standards state that all students in all grades should learn "to use the language of mathematics to express mathematical ideas precisely." According to the NCTM,

> *"As students articulate their mathematical understanding in the lower grades, they begin by using everyday, familiar language. This provides a base on which to build a connection to formal mathematical language. Teachers can help students see that some words that are used in everyday language, such as similar, factor, area, or function, are used in mathematics with different or more-precise meanings. This observation is the foundation for understanding the concept of mathematical definitions. It is important to give students experiences that help them appreciate the power and precision of mathematical language. Beginning in the middle grades, students should understand the role of mathematical definitions and should use them in mathematical work. Doing so should become pervasive in high school. However, it is important to avoid a premature rush to impose formal mathematical language; students need to develop an appreciation of the need for precise definitions and for the communicative power of conventional mathematical terms by first communicating in their own words."*

Although many college students say mathematics is a language, usually "the language of science," few of them seem to fully understand the implications of that statement, especially as it relates to mathematical definitions. In a research project sponsored by Oregon's CETP program, and also in separate research by the first author, we found that some of the difficulties undergraduates have in upper division mathematics can be explained by their failure to categorize and utilize mathematical definitions properly. (Further discussion of these findings may be found in Edwards & Ward, 2004.)

Therefore, as a result of our research and in keeping with the NCTM recommendations, the authors have initiated teaching practices that emphasize the proper categorization and use of mathematical definitions. In particular, we teach the difference between mathematical definitions and the definitions of "everyday" language. The purpose of this paper is to describe the changes in our teaching practices that have evolved as a result of our work in definitions.

NEW PRACTICES

The following innovations have been tried by one or both authors, primarily in post-calculus mathematics courses taken by pre-service secondary mathematics teachers. However, a modified version of the introduction to the philosophy of definitions was used by one of the authors in a course for pre-service elementary and middle school teachers.

A Brief Introduction to the Philosophy of Definitions

Typically in college mathematics courses, it seems to be assumed that students know that mathematical definitions are different from "everyday" definitions, or that if students do not know about the special nature of these definitions, they will "pick it up" in the natural course of studying mathematics. It is probably true that many undergraduate mathematics majors do eventually understand the nature of mathematical definitions, but we believe that the concept is far too important to leave to chance.

In our courses, we discuss the philosophical categorization of definitions. Natural or "everyday" language definitions are in the *extracted* category, so called because they are extracted from instances of actual usage (Landau, 2001). These definitions have a truth value, that is, the definitions may or may not have been accurately reported.

Definitions in mathematics are quite different. They are *stipulated*, meaning they do not report usage, but rather specify usage. In the words of Robinson (1954), a mathematical definition of a term is "not a historical description of what has been meant by [the term] in the past or is commonly meant by it now" as is the case with extracted definitions. Rather, the mathematical definition is "an announcement of what is going to be meant by [the term] in the present work, or a request to the reader to take it in that sense" (Robinson, 1954, p. 59).

As a result, stipulated definitions lack a truth value. They are not right or wrong, provable or disprovable. They simply constitute an agreement between writer and reader concerning the meaning of a term in the current context. There is an analogy in the law, where documents may begin with a stipulation of the meaning of terms such as "lender" or "borrower."

Furthermore, unlike extracted definitions, mathematical definitions cannot be reliably learned by repeated exposure to instances of the definition, so-called ostensive definition. We observe toddlers learning natural language definitions ostensively all the time.

Young children can often point to a picture of a bird in response to the question "Where's the bird?" before they can say the word "bird." They have evidently learned a meaning of "bird" by having others repeatedly point out instances of birds to them. Stipulated definitions generally cannot be accurately acquired in that way.

Some students seem to instinctively understand the nature of mathematical definitions. Others, however, show signs of confusion. They routinely paraphrase definitions in ways not equivalent to the original. They avoid memorizing mathematical definitions in favor of something akin to ostensive definition. By introducing them to the nature of mathematical definitions as outlined above, we believe some confused students will be nudged onto the path of proper understanding.

As another example, we sometimes engage the students in the process of stipulating a definition. We do not mean trying to get the students to guess some widely accepted mathematical definition. Rather we mean finding a situation in which something is true about a certain class of objects, then letting students stipulate a name for that class and prove theorems about the class. This process models the way mathematicians make definitions in their research work.

To illustrate: Consider the class of rectangles where the sides have whole number lengths and where the perimeter is 12. Students could stipulate that such a rectangle is called an "even-dozen rectangle" (or a "perimeter-12 rectangle" or whatever they choose). They could then be led to discover and prove the theorem: If R is an even-dozen rectangle, then R measures 1-by-5, 2-by-4 or 3-by-3. A less trivial example, about the class of triangles on a sphere for which the Side-Angle-Side congruence criterion holds, is mentioned in Edwards & Ward (2004, p. 420-421).

Instruction in the Use of Mathematical Definitions

Here we use a simple cognitive model from the mathematics education literature (Vinner, 1991) in which each mathematical concept has an associated concept definition and concept image. In our version of the model, which we employ in our classes, the concept definition is simply the stipulated definition of that concept. The concept image is comprised of the various intuitions associated with the concept, for instance, the mental images, prototypical examples, experiences and analogies.

Vinner uses the following simple figures to illustrate the ways in which a mathematical task, like proving a theorem, may be done. In Figures 1-3, the task is completed in a mathematically acceptable way. Notice the key feature is that the completion of the task is ultimately based on the concept definition. Figure 4, on the other hand, describes task completion which is not rigorous and, therefore, is not mathematically acceptable.

To summarize, although the concept image may and should play a part in the completion of the task, the concept definition is the foundation upon which the result must rest. The best research mathematicians have powerful concept images which guide their work, but even they must give rigorous proofs based on the stipulated definitions or else they have nothing but informed conjectures. "No proof, no mathematics" is the oft-repeated characterization of mathematics.

After introducing Vinner's model to our students we might from time to time look at proofs of theorems or solutions of problems and decide which figure best models the process by which we arrived

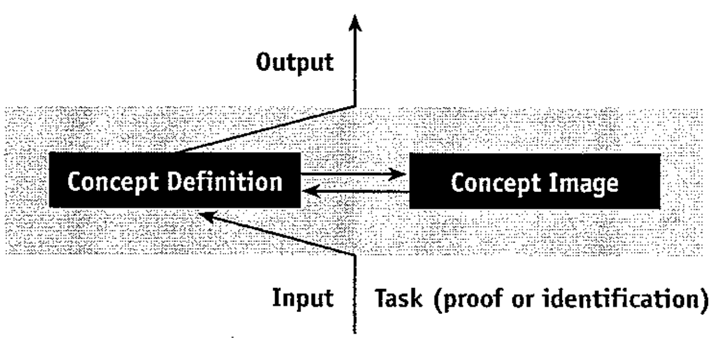

Figure 1. Interplay between definition and image [5].

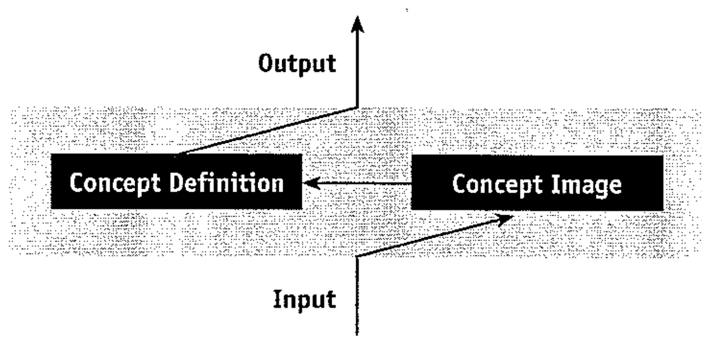

Figure 2. Deduction following intuitive thought [5].

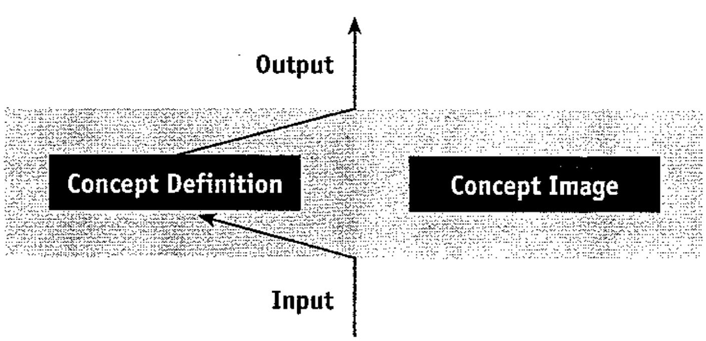

Figure 3. Purely formal deduction [5].

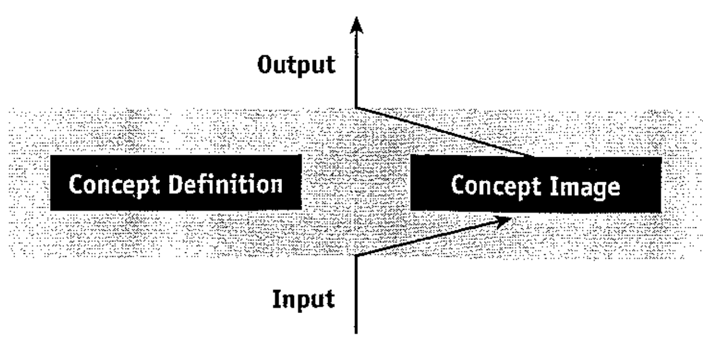

Figure 4. Intuitive response [5].

at the proof or solution. We also have become more careful, in all levels of instruction, to clearly differentiate the concept image from the stipulated concept definition. Heretofore, we might have stated a definition, then, practically in the same breath, said "But what it really means is . . . " and then gone on to give pictures or other insights which are part of our own concept image. In some lower division classes, we might then actually have done some problem solving based on the concept image rather than the definition. We now believe this blurring of concept image and concept definition is a mistake. As a result, we are much more careful to distinguish when we are doing proper, rigorous, mathematics and when we are working in the non-rigorous realm of concept image.

By presenting the notions of concept definition and concept image and by discussing the figures as above, we hope to instill in our future teachers a clearer sense of the proper use of mathematical definitions, in other words, a clearer sense of the role definitions play in mathematics. We hope it will help them to succeed in their upper-division math courses and, more importantly, will enable them to communicate mathematically with perspective and deeper understanding.

LESSONS LEARNED

Our research suggests that it is not just misunderstanding the content of mathematical definitions which causes difficulties for undergraduates but, more fundamentally, it is also misunderstanding their nature and use. Some of that misunderstanding may have come from or at least may have been reinforced by teachers who themselves did not clearly grasp the role of definitions in mathematics.

The lesson here is that we ought to give explicit instruction on these issues to pre-service teachers. We have found that many undergraduates seem to grasp fairly readily the notions of stipulated and extracted definitions and the concept image/concept definition model. Quite a few adopt that terminology in class discussions. We, therefore, have reason to hope these ideas become part of the well of conceptual understanding from which these future teachers draw in order to improve and to inform their teaching.

Our goal is that they will beware of the subtle ways in which they might be misleading their students and also alert to the subtle ways in which they might lead their students to "an appreciation of the need for precise definitions and for the communicative power of conventional mathematical terms" (NCTM 2004, p. 63).

NEXT STEPS

From our perspective as researchers, we see at least two areas for investigation. First, the effectiveness of our new practices needs to be assessed. Are students' understanding and use of mathematical definitions actually improved by the activities we use? Do our methods have any effect on the way the pre-service teachers eventually teach? Second, understanding mathematical definitions among pre-service and in-service teachers, both secondary and elementary, needs to be researched more thoroughly. It would be interesting to conduct studies with similar methodologies involving pre-service or in-service teachers who have already completed most of their mathematical preparation. Such studies might suggest additional teacher preparation strategies.

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