

ON MINIMAL NON- p -CLOSED GROUPS AND RELATED PROPERTIES

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ABSTRACT. Let p be a prime. A group is called p -closed if it has a normal Sylow p -subgroup and it is called p -exponent closed if the elements of order dividing p form a subgroup. A group is minimal non- p -closed if it is not p -closed but its proper subgroups and homomorphic images are. Similarly, a group is called minimal non- p -exponent closed if it is not p -exponent closed but all its proper subgroups and homomorphic images are. In this paper we characterize finite minimal non- p -closed groups and investigate the relationship between them and minimal non- p -exponent closed groups. In particular, we show that every minimal non- p -closed group is non- p -exponent closed and that minimal non- p -closed groups and simple minimal non- p -exponent closed groups have cyclic Sylow p -subgroups. Furthermore, given a prime p , we describe non- p -exponent closed groups of smallest order and we show that they coincide with non- p -closed groups of smallest order.

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1. INTRODUCTION

Let E be a group theoretic property. We say that a group G is a minimal non- E -group if G is not an E -group but proper subgroups and proper homomorphic images of G are E -groups. Already in 1903, Miller and Moreno [12] characterized finite minimal non- E -groups for the property $E = \text{abelian}$. Their investigations were completed by Rédei [15]. In [12] and [15], the same three authors considered the class of non-abelian groups with every proper subgroup abelian. As it turned out, this class coincides with the class of minimal non-abelian groups. Minimal non- E -groups have been investigated for various properties such as nilpotency, solvability, and others. We refer to [6] for further details.

The topic of our investigation is the class of minimal non- E -groups, where E is either the class of finite p -closed groups or the class of finite p -exponent closed groups. In [1], Reinhold Baer defined the following property.

Definition 1.1. *Let Σ be a set of primes. A group G is called Σ -closed if the set of all elements of G whose order has no prime divisor outside of Σ is a subgroup of G .*

In particular, a group is called p -closed if it is Σ -closed for $\Sigma = \{p\}$. Note that a finite group is p -closed if and only if it has a normal Sylow p -subgroup. In particular, all abelian groups are p -closed.

In [9], n -exponent closed groups are defined as follows.

Definition 1.2. *Let G be a group and let n be a positive integer. Consider the set $G[n] = \{g \in G : g^n = 1\}$ and let $G_n = \langle G[n] \rangle$ be the subgroup of G generated by $G[n]$. We say that G is n -exponent closed provided $G[n] = G_n$.*

Clearly, an abelian group is n -exponent closed for any n . In particular, the concept of an n -exponent closed group has been studied in the case when G is a p -group and n is a power of p . In this case, it is customary to write $\Omega_{\{i\}}(G)$ or $\Lambda_i(G)$ for $G[p^i]$ and $\Omega_i(G)$ for G_{p^i} .

The classes of p -closed groups and p -exponent closed groups appear to have some similarities, but neither is a subclass of the other. For example, any finite p -group is p -closed but there is a wealth of finite p -groups which are not p -exponent closed. An example of smallest order is the wreath product $W = C_p \wr C_p$ of a cyclic group of order p with itself. The group W has order p^{p+1} , exponent p^2 , and it is generated by elements of order p . Thus any finite group having W as a normal Sylow p -subgroup is p -closed but not p -exponent closed. On the other hand, if q is a prime divisor of $2^p - 1$, then the group

$$T = T(p, q) = \langle a, b \mid a^q = b^{p^2} = 1, b^{-1}ab = a^2 \rangle$$

is not p -closed but it is p -exponent closed, since $T[p] = \langle b^p \rangle$ is the center of T .

In light of the above remarks, it comes somewhat as a surprise that there is a strong connection between minimal non- p -closed groups and minimal non- p -exponent closed groups. As we will show, any minimal non- p -closed group and any minimal non- p -exponent closed group is either solvable or simple (Theorem 3.1). In the solvable case,

any minimal non- p -closed group is also minimal non- p -exponent closed, while a solvable minimal non- p -exponent closed group is either minimal non- p -closed or it is a p -group (Theorems 3.1 and 4.1). The minimal non- p -exponent closed p -groups are 2-generated, have class at least p , exponent p^2 and a cyclic center. A full classification of these p -groups seems out of reach at the moment.

Looking at simple groups, we show that every simple minimal non- p -closed group is also a minimal non- p -exponent closed group (Theorem 3.5). We do not know if the converse always holds. Looking carefully at the techniques of [10], where the simple minimal non-7-closed groups are classified, one can verify that the converse holds for $p \leq 7$ (section 3). The proof relies on the classification of finite simple groups and it is crucial for the argument that p is small. Obtaining a similar classification for all primes p will require substantially new ideas. However, we show that simple minimal non- p -closed groups have cyclic Sylow p -subgroups with pairwise trivial intersection (Theorem 3.5; the fact that the Sylow p -subgroups are cyclic was established independently in [17]). We also prove that simple minimal non- p -exponent closed groups have cyclic Sylow p -subgroups, but whether they have pairwise trivial intersection remains open (Theorem 4.3). Incidentally, only the results in this paragraph rely on the classification of finite simple groups.

To introduce our final results, something should be said about what led us to the investigation of these two classes of groups. The inspiration came from a paper by Desmond MacHale [11]. In this paper 47 properties of groups are considered and for each property MacHale asks for groups of smallest order which do not have the property. The first property in [11] concerns groups in which the set of all squares of elements is a subgroup. As can be easily seen, the alternating group A_4 and the group $T(2, 3)$ mentioned above are exactly the groups of smallest order which do not have this property. A natural extension of this result is to consider groups in which the set of all n -th powers of elements is a subgroup. Such groups are called n -power closed [9]. In [8], groups of smallest order which are not n -power closed have been determined for any n not divisible by 16. A dual problem is to determine groups of smallest order which are not n -exponent closed. To solve this problem in the case n is a prime was the original motivation for our investigation. We show that G is a non- p -closed group of smallest order if and only if it is a non- p -exponent closed group of smallest order and

then G is either solvable or isomorphic to $PSL(2, p)$ (Theorem 5.3). As a corollary of our description of non- p -exponent closed groups of smallest order we get the following formula for their order $f(p)$:

$$f(2) = 6 \quad \text{and} \quad f(p) = \min\{p(kp + 1), \frac{1}{2}p(p^2 - 1)\},$$

where p is an odd prime and k is the smallest positive integer such that $kp + 1$ is a prime power (Corollary 5.4). This formula leads us to an old problem in analytic number theory about the smallest prime in an arithmetic progression, which we discuss in more detail in the last section.

2. A FAMILY OF FINITE SOLVABLE GROUPS

In this section we introduce a family of finite solvable groups which will play a major role in the next section, where we characterize minimal non- p -closed groups.

Let p and q be distinct primes. Denote by $e_p(q)$ the order of q modulo p . Let \mathbb{F} be a finite field with $q^{e_p(q)}$ elements. The multiplicative group \mathbb{F}^\times of \mathbb{F} is cyclic of order $q^{e_p(q)} - 1$. Since p divides $q^{e_p(q)} - 1$, the group \mathbb{F}^\times has a unique subgroup S of order p . The group S acts on the additive group F of \mathbb{F} by multiplication. Denote by $U(p, q)$ the semidirect product $F \rtimes S$. We have the following result.

Proposition 2.1. *Let p and q be distinct primes and let F , S , and $e_p(q)$ be as defined above. Then the following hold:*

- (i) *The group S acts irreducibly on F ;*
- (ii) *Let P be a cyclic group of order p . If A is an elementary abelian q -group of order q^m on which P acts irreducibly, then $m = e_p(q)$ and the semidirect product $A \rtimes P$ is isomorphic to $U(p, q)$.*
- (iii) *$U(p, q)$ is minimal non- p -closed and minimal non- p -exponent closed.*

Proof. Let \mathbb{F}_q be the prime subfield of \mathbb{F} . Let u be a generator of S . We claim that $\mathbb{F}_q[u] = \mathbb{F}$. In fact, $\mathbb{F}_q[u]$ is a field with q^s elements for some $s \leq e_p(q)$. Thus $u^{q^s-1} = 1$ and therefore $p|q^s - 1$. It follows from the definition of $e_p(q)$ that $e_p(q)|s$. Consequently, $s = e_p(q)$ and $\mathbb{F} = \mathbb{F}_q[u]$. Any S -invariant subgroup of F is invariant under multiplication by any element of $\mathbb{F}_q[u]$. Since $\mathbb{F}_q[u] = \mathbb{F}$, it follows that $\{0\}$ and F are the only S -invariant subspaces of F . This shows that S acts irreducibly on F .

Suppose now that P is a cyclic group of order p and A is an elementary abelian q -group of order q^m on which P acts irreducibly. Let c be a generator of P . Then A is a simple module over the polynomial ring $\mathbb{F}_q[x]$, where x acts on A as c . It follows that A can be identified with $\mathbb{F}_q[x]/J$ for some maximal ideal J of $\mathbb{F}_q[x]$. Since the ideal J is maximal, the ring $\mathbb{F}_q[x]/J$ is a field which we denote by \mathbb{F} . Thus A is identified with the additive group F of \mathbb{F} . Moreover, the image of x in \mathbb{F} is an element of order p and the action of P on A is identified with the action of S on F by multiplication, where S is the cyclic subgroup of \mathbb{F}^\times of order p . Let u be a generator of S . The additive group of the subfield $\mathbb{F}_q[u]$ of \mathbb{F} is an S -invariant subgroup of F . Since F is irreducible, we have $\mathbb{F}_q[u] = \mathbb{F}$. It follows that m is the smallest positive integer such that $u^{q^m} = u$, i.e. $u^{q^m-1} = 1$. Equivalently, m is the smallest positive integer such that p divides $q^m - 1$, i.e. $m = e_p(q)$. This proves that $A \rtimes P$ is isomorphic to $U(p, q)$.

Since Sylow p -subgroups of $U(p, q)$ are cyclic of order p and not normal, the group $U(p, q)$ does not have any non-trivial normal p -subgroups. Hence it is neither p -closed nor p -exponent closed. It follows from (i) that every proper subgroup of $U(p, q)$ is abelian and F is the only non-trivial proper normal subgroup. Now (iii) follows easily. \square

3. MINIMAL NON- p -CLOSED GROUPS

In this section we characterize minimal non- p -closed groups. Recall that if E is a property of finite groups then a minimal non- E -group is any finite group G which does not have the property E but the proper subgroups and proper quotients of G have property E . For some properties E , e.g. abelian, if all proper subgroups of a group have property E , then all proper quotients of this group also have property E . This is not the case for the property of being p -closed. Consider the group $T = T(p, q)$ introduced in Section 1. All proper subgroups of T are abelian, hence are p -closed, but the quotient of T by its center is not p -closed. Non- p -closed groups with all proper subgroups p -closed have been called *inner- p -closed* (see e.g. [10]). Thus $T(p, q)$ is inner- p -closed but not a minimal non- p -closed group.

The following theorem describes minimal non- p -closed groups.

Theorem 3.1. *A finite group G is a minimal non- p -closed group if and only if it satisfies one of the following conditions:*

- (i) *G is a simple group which is a minimal non- p -closed group;*
- (ii) *G is isomorphic to $U(p, q)$ for some prime q different from p .*

Proof. Let G be a minimal non- p -closed group which is not simple and let P be a Sylow p -subgroup of G . Let A be a minimal normal subgroup of G , then A is a proper subgroup and the quotient G/A is p -closed. Thus G/A has a normal Sylow p -subgroup. It follows that all Sylow p -subgroups of G are contained in AP . Since G is not p -closed, neither is AP . Thus $AP = G$, since all proper subgroups of G are p -closed. It follows that G/A is a p -group.

Note that A is a direct product of copies of a simple group H . We claim that p does not divide the order of H . In fact, suppose that p divides $|H|$. Then, since H is simple and p -closed, H must be abelian. Thus A is an elementary abelian p -group. It follows that G is a p -group, a contradiction. We conclude that $|A|$ is prime to p . Thus $G = AP$ is a semidirect product. For any maximal subgroup M of P , the subgroup AM of G is p -closed and M is a Sylow p -subgroup of AM . It follows that M is normal in AM . Thus A normalizes M . Since maximal subgroups of P are normal in P , we see that M is normal in G . Now if M is non-trivial, then G/M is p -closed. Since M is a p -group, it follows that G is p -closed, a contradiction. Thus all maximal subgroups of P must be trivial, hence P is cyclic of order p .

Suppose that H is non-abelian. For any prime divisor q of H , the group P acts by conjugation on the set of Sylow q -subgroups of A . Since the number of such subgroups divides $|A|$, it is prime to p and therefore the action has a fixed point. In other words, P normalizes a Sylow q -subgroup S_q of A , which is also a Sylow q -subgroup of G . Since S_q is a proper subgroup of A , the group S_qP is a proper subgroup of G , hence it is p -closed. Thus P is normal in S_qP , i.e. S_q normalizes P . It follows that the normalizer of P contains a Sylow q -subgroup of G for every prime q dividing $|G|$. This however means that P is normal in G , a contradiction. Thus H must be abelian. It follows that A is an elementary abelian q -group for some prime q different from p . Being a minimal normal subgroup of G , the group A is an irreducible P -module. By

Proposition 2.1, the group G is isomorphic to $U(p, q)$. Conversely, the groups $U(p, q)$ are minimal non- p -closed groups by Proposition 2.1. \square

The question arises if there are non-abelian simple groups which are minimal non- p -closed, and if there exist such groups, can they be classified? It follows from Dickson's Theorem (see e.g. [6, II.8.2]) that for any prime $p > 3$ the group $PSL(2, p)$ is a simple minimal non- p -closed group. For small primes p the classification of finite simple groups can be employed to list all simple minimal non- p -closed groups. The main idea is a trivial observation that simple minimal non- p -closed group cannot have proper simple subgroups of order divisible by p . When p is small, the groups minimal among the simple groups of order divisible by p are easy to list. Using known properties of these groups one verifies which among them are minimal non- p -closed. The details have been worked out for $p \leq 7$. From the classification in [10] and the related results quoted therein, we get the following description of simple minimal non- p -closed groups for $p \leq 7$:

- (i) all minimal non-2-closed groups are solvable;
- (ii) simple minimal non-3-closed groups are exactly the groups $PSL(2, 2^q)$, q an odd prime;
- (iii) simple minimal non-5-closed groups are exactly the groups $PSL(2, 5)$ and $Sz(2^q)$, q an odd prime;
- (iv) simple minimal non-7-closed groups are exactly the groups $PSL(2, 7)$, $PSL(2, q)$ for any prime $q \equiv -1 \pmod{7}$, and $PSL(2, q^3)$ for any prime $q \equiv 3$ or $5 \pmod{7}$.

A similar classification for all $p > 7$ seems out of reach in general. However, we show that simple minimal non- p -closed groups have cyclic Sylow p -subgroups with pairwise trivial intersection. The proof is based on the classification of simple groups with a strongly p -embedded proper subgroup. Recall that a subgroup H of a group G is called *strongly p -embedded* if for any non-trivial subgroup Q of some Sylow p -subgroup of G , the normalizer of Q in G is contained in H .

Proposition 3.2. *Let G be a simple minimal non- p -closed group.*

- (i) *Any two distinct Sylow p -subgroups of G have trivial intersection.*
- (ii) *The normalizer of a Sylow p -subgroup of G is a strongly p -embedded subgroup.*

Proof. Let P_1, P_2 be two distinct Sylow p -subgroups of G such that the order of $Q = P_1 \cap P_2$ is largest possible. It suffices to show that Q is the trivial group. Suppose otherwise and consider the normalizer N of Q in G . Since G is simple, N is a proper subgroup of G . It follows that N has a normal Sylow p -subgroup S , which is contained in a Sylow p -subgroup P of G . Now the normalizer Q_i of Q in P_i is a p -group strictly containing Q and contained in N , $i = 1, 2$. Thus $Q_i \subseteq S \subseteq P$, hence $P \cap P_i$ has order larger than the order of Q . This however implies that $P = P_i$ and consequently $P_1 = P_2$, a contradiction. This completes the proof of (i).

Consider now a Sylow p -subgroup P of G and a non-trivial subgroup Q of P . If $x \in G$ normalizes Q , then Q is contained in both P and xPx^{-1} and therefore $P = xPx^{-1}$ by (i). In other words, the normalizer of Q is contained in the normalizer of P . Thus (ii) holds. \square

In [16], a classification of simple groups of Lie type with a strongly p -embedded proper subgroup has been obtained (see, in particular, Theorem 7 therein). Using the classification of finite simple groups and some facts about the sporadic simple groups (contained, for example, in [3]), one can extend this to a list of all simple groups with a strongly p -embedded proper subgroup (see [5, Th. 4.249]). Note that both in [16] and [5] the classification is obtained for groups with simple generalized Fitting subgroup. Restricting to simple groups, we obtain the following result.

Theorem 3.3. *Let G be a finite simple group with a strongly p -embedded proper subgroup. Then one of the following conditions holds:*

- (i) G has cyclic Sylow p -subgroups;
- (ii) $G \cong PSL(2, q), PSU(3, q), Sz(q), {}^2G_2(q)$, or A_{2p} , and q is a power of p ;
- (iii) $p = 3$ and $G \cong PSL(3, 4)$, or M_{11} ;
- (iv) $p = 5$ and $G \cong {}^2F_4(2)'$, Mc , or $M(22)$;
- (v) $p = 11$ and $G \cong J_4$.

Recall that a finite group is quasi-simple if it is perfect and its quotient modulo the center is simple. Theorem 3.3 allows us to prove the following result.

Corollary 3.4. *Let p be an odd prime and let G be a finite simple group with a strongly p -embedded proper subgroup such that the simple quotient of any proper quasi-simple subgroup of G has order prime to p . Then G has cyclic Sylow p -subgroups.*

Proof. The group G is one of the groups described in (i)-(v) of Theorem 3.3. The groups in (iii)-(v) are eliminated by using the Atlas [3] to verify that they have proper non-abelian simple subgroups of order divisible by p . If G is one of the groups of (ii), then it is isomorphic to $PSL(2, p)$, $PSU(3, p)$, or $p = 3$ and $G \cong {}^2G_2(27)$. For otherwise, it clearly has a proper non-abelian simple subgroup of order divisible by p (note that $Sz(q)$ is eliminated since $p \neq 2$). Using the Atlas, we see that ${}^2G_2(27)$ has proper non-abelian simple subgroups of order divisible by 3, hence it is not isomorphic to G . For $p \geq 5$, the group $PSU(3, p)$ has a proper subgroup isomorphic to $SL(2, p)$ ([13]). From the Atlas we see that the group $PSU(3, 3)$ has a proper subgroup isomorphic to $PSL(2, 7)$. It follows that G can not be isomorphic to $PSU(3, p)$. This shows that if G is one of the groups of (ii), then $G \cong PSL(2, p)$. Since $PSL(2, p)$ has cyclic Sylow p -subgroups, the result follows. \square

We now obtain the following result.

Theorem 3.5. *Let G be a minimal non- p -closed group. Then G has cyclic Sylow p -subgroups with pairwise trivial intersections. In particular, G is minimal non- p -exponent closed.*

Proof. According to Theorem 3.1, G is either simple or isomorphic to $U(p, q)$ for some q . In the latter case, the first part of the theorem is clear. Suppose then that G is simple. We claim that p is odd in this case. Indeed, suppose that $p = 2$ and let x be an element of order 2. By the Baer-Suzuki Theorem [4, Theorem 3.8.2], there is a conjugate y of x in G such that the group $\langle x, y \rangle$ is not a 2-group. Thus $\langle x, y \rangle$ is not 2-closed and being a dihedral group, it must be a proper subgroup of G . This shows that G is not a minimal non-2-closed group. Thus p is odd and the first part of the theorem follows from Proposition 3.2 and Corollary 3.4.

To justify the last part, note that neither a simple group of order divisible by p nor $U(p, q)$ is p -exponent closed. Since G is minimal non- p -closed, any proper subgroup H of G has a cyclic, normal Sylow p -subgroup. Hence H is p -exponent closed. \square

4. MINIMAL NON- p -EXPONENT CLOSED GROUPS

In this section we characterize minimal non- p -exponent closed groups. It turns out that these groups are closely related to the minimal non- p -closed groups investigated in the preceding section. We point out however that, unlike the class of p -closed groups, the class of p -exponent closed groups is not quotient closed, as can be seen from the groups $T(p, q)$ introduced in Section 1. The main theorem of this section is very similar to Theorem 3.1 except that the list of minimal non- p -exponent closed groups includes some p -groups.

Theorem 4.1. *A finite group G is a minimal non- p -exponent closed group if and only if it satisfies one of the following conditions:*

- (i) G is a simple group which is a minimal non- p -exponent closed group;
- (ii) G is a p -group which is a minimal non- p -exponent closed group. In this case G is 2-generated, has class at least p , exponent p^2 , and cyclic center. In particular, $|G| \geq p^{p+1}$;
- (iii) G is isomorphic to $U(p, q)$ for some prime $q \neq p$.

Proof. Let G be a minimal non- p -exponent closed group which is not simple. There are two elements $x, y \in G$ of order p such that the order of xy is neither 1 nor p . The minimality of G ensures that G is generated by x and y . Let A be a minimal normal subgroup of G . Since G is not simple, the subgroup A is a non-trivial proper subgroup. Let $\bar{G} = G/A$. Again by minimality of G , the group \bar{G} is p -exponent closed. Since \bar{G} is p -exponent closed, the set $\bar{G}[p]$ is a subgroup of \bar{G} . Note that both xA and yA belong to $\bar{G}[p]$. Since G is generated by x and y , we conclude that $\bar{G} = \bar{G}[p]$. In other words, G/A is a group of exponent p .

Note that A is a direct product of copies of a simple group H . Suppose that p divides $|H|$. The group H , being isomorphic to a proper subgroup of G , is a p -exponent closed group. Since H is simple, it must be abelian. It follows that A is an elementary abelian p -group and therefore G is a p -group. Consequently, the group A contains a non-trivial element c of order p which is central in G . Since A is a minimal normal subgroup of G , we have $A = \langle c \rangle$. Thus A is cyclic of order p and it is central in G . It follows that G has exponent p^2 and therefore xy has order p^2 . Since G/A has exponent p , we

have $(xy)^p$ is a nontrivial element of A , hence $A = \langle (xy)^p \rangle$. This together with our assumption that A is an arbitrary minimal normal subgroup of G shows that A is the unique minimal normal subgroup of G . Consequently, A coincides with the center of G . Since p -groups of class less than p are regular and regular p -groups are p -exponent closed, the class of G is at least p . Clearly a group of class p has order at least p^{p+1} . This establishes (ii) of the theorem.

Suppose now that p does not divide the order of H . Then $|A|$ is prime to p and G/A is a p -group. It follows that G is isomorphic to a semidirect product $A \rtimes G/A$. In particular, the Sylow p -subgroups of G have exponent p . Note that for groups with a Sylow p -subgroup of exponent p , to be p -closed is the same as to be p -exponent closed. It follows that G is a minimal non p -closed group. By Theorem 3.1, the group G is isomorphic to a group of the form $U(p, q)$. Conversely, the groups $U(p, q)$ are minimal non- p -exponent closed by Proposition 2.1. \square

Two questions naturally arise: what are the minimal non- p -exponent closed p -groups and what are the simple minimal non- p -exponent closed groups? For the first question, consider the wreath product $W = C_p \wr C_p$, where C_p is a cyclic group of order p . The group W is a minimal non- p -exponent closed group of order p^{p+1} . Indeed, this group has exponent p^2 and is generated by elements of order p (see [6, III.10.2]). Every proper subgroup or quotient group of W has order at most p^p hence it is a p -exponent closed group. We do not know of any other construction of minimal non- p -exponent closed p -groups. Note that Theorem 4.1(ii) implies that for each p there is only a finite number of p -groups which are minimal non- p -exponent closed. It would be nice to have an explicit description of all such groups.

As for the second question, it follows from Dickson's Theorem (see e.g. [6, II.8.2]) that for any prime $p > 3$, the group $PSL(2, p)$ is a simple minimal non- p -exponent closed group. By Theorem 3.5, every simple minimal non- p -closed group is also a minimal non- p -exponent closed group. We do not know if the converse holds for all p . Using the methods of [10], one can verify that this is indeed true for $p \leq 7$. In general, we will show that simple minimal non- p -exponent closed groups have cyclic Sylow p -subgroups, but we have not been able to show that different such subgroups have trivial intersection. The key is the following observation.

Proposition 4.2. *Let G be a simple minimal non- p -exponent closed group. Then the following hold:*

- (i) *If P_1, P_2 are different Sylow p -subgroups of G , then either they have trivial intersection or $\Omega_1(P_1) = \Omega_1(P_2)$;*
- (ii) *If P is a Sylow p -subgroup of G , then the normalizer of $\Omega_1(P)$ is a strongly p -embedded subgroup.*

Proof. Let P_1, P_2 be two different Sylow p -subgroups of G such that $\Omega_1(P_1) \neq \Omega_1(P_2)$ and the group $Q = \Omega_1(P_1) \cap \Omega_1(P_2)$ has largest possible order. It suffices to show that Q is the trivial group. Suppose otherwise and consider the normalizer N of Q in G . Since G is simple, N is a proper subgroup of G . The normalizer Q_i of Q in $\Omega_1(P_i)$ contains Q properly and is contained in N , $i = 1, 2$. Let $B = N[p]$. Thus B is a normal p -subgroup of N and it contains Q_i , $i = 1, 2$. Let P be a Sylow p -subgroup of G containing B . Then $B \subseteq \Omega_1(P)$. It follows that the order of $\Omega_1(P) \cap \Omega_1(P_i)$ is larger than the order of Q . This however implies that $\Omega_1(P) = \Omega_1(P_i)$ and consequently $\Omega_1(P_1) = \Omega_1(P_2)$, a contradiction. This completes our proof of (i).

Consider now a Sylow p -subgroup P of G and a non-trivial subgroup Q of P . If $x \in G$ normalizes Q , then Q is contained in both P and xPx^{-1} and therefore $\Omega_1(P) = \Omega_1(xPx^{-1}) = x\Omega_1(P)x^{-1}$ by (i). In other words, the normalizer of Q is contained in the normalizer of P . \square

The last proposition combined with Corollary 3.4 yields now the following result. The proof is similar to the proof of Theorem 3.5.

Theorem 4.3. *Let G be a simple minimal non- p -exponent closed group. Then G has cyclic Sylow p -subgroups.*

5. NON- p -EXPONENT CLOSED GROUPS OF SMALLEST ORDER.

We have not been able to characterize all simple groups which are minimal non- p -closed or minimal non- p -exponent closed groups. We can however describe such groups of smallest possible order. The key is the following theorem, conjectured by E. Artin and proved by R. Brauer and W. F. Reynolds [2].

Theorem 5.1. *Let G be a non-abelian simple group of order $|G|$ divisible by a prime p such that $|G| < p^3$. Then $p > 3$ and either G is isomorphic to $PSL(2, p)$ or $p = 2^n + 1$ is a Fermat prime and G is isomorphic to $PSL(2, 2^n)$.*

Corollary 5.2. *Let p be a prime and let G be a finite non-abelian simple group of smallest possible order divisible by p . If $p > 3$, then G is isomorphic to $PSL(2, p)$. In particular, G is both a minimal non- p -closed group and a minimal non- p -exponent closed group. If $p \in \{2, 3\}$, then G is isomorphic to the alternating group A_5 .*

Proof. The conclusion is clear for $p \in \{2, 3\}$. Note that for $p \in \{2, 3\}$, the group A_5 is neither a minimal non- p -closed group nor a minimal non- p -exponent closed group.

Suppose that $p > 3$. Since p divides $|PSL(2, p)|$, we have $|G| \leq |PSL(2, p)| < p^3$ and Theorem 5.1 applies. If $p = 5$, then $G \cong PSL(2, 5) \cong PSL(2, 2^2)$. If $p > 5$ and $p = 2^n + 1$ is a Fermat prime, then

$$|PSL(2, 2^n)| = (2^{2n} - 1)2^n = p(p - 2)(p - 1) > \frac{1}{2}p(p^2 - 1) = |PSL(2, p)|.$$

The first part of the corollary is now an immediate consequence of Theorem 5.1. The second part follows immediately from Dickson's Theorem [6, II.8.27]. \square

For a prime p , let $n(p) = q^m$ be the smallest prime power congruent to 1 modulo p , where $q = q(p)$ is a prime. Clearly, $n(p) = kp + 1$, where k is the smallest positive integer such that $kp + 1$ is a power of a prime. The following theorem describes non- p -closed groups of smallest order and non- p -exponent closed groups of smallest order.

Theorem 5.3. *Let p be a prime. The following conditions for a finite group G are equivalent:*

- (i) G is a non- p -closed group of smallest order;
- (ii) G is a non- p -exponent closed group of smallest order;
- (iii) Either $\frac{1}{2}(p^2 - 1) \leq n(p)$ and G is isomorphic to $PSL(2, p)$ or $\frac{1}{2}(p^2 - 1) > n(p) = q^m$ and G is isomorphic to $U(p, q)$.

Proof. Since $|PSL(2, p)| < p^{p+1}$, a non- p -exponent closed group of smallest order is not a p -group by Theorem 4.1(ii). Note that $PSL(2, 2) \cong U(2, 3) \cong S_3$, $PSL(2, 3) \cong U(3, 2) \cong A_4$, and $\frac{1}{2}(p^2 - 1) \neq n(p)$ for $p > 3$. These observations together with Theorems 3.1, 4.1, and Corollary 5.2 easily imply the theorem. \square

As a corollary to Theorem 5.3 we get an answer to the question which led us into the investigations of this paper: given a prime p , what is the smallest order of a non- p -exponent closed group?

Corollary 5.4. *Let p be a prime and let $f(p)$ denote the smallest order of a non- p -exponent closed group. Then*

$$f(p) = \begin{cases} 6, & \text{if } p = 2; \\ \min\{p \cdot n(p), \frac{1}{2}p(p^2 - 1)\}, & \text{if } p \text{ is odd.} \end{cases}$$

Given a prime $p > 3$, it is natural to ask which of the groups $PSL(2, p)$, $U(p, q(p))$ has smaller order. In other words, we would like to know which of the numbers $n(p)$ and $\frac{1}{2}(p^2 - 1)$ is smaller. With the help of a computer we checked that among the first 2000 primes only $p = 19$ satisfies $n(p) > \frac{1}{2}(p^2 - 1)$. It seems reasonable to conjecture that 19 is the only such prime. This leads us to the following question: given an odd prime p different from 19, is there a prime power which is congruent to 1 modulo p and does not exceed $\frac{1}{2}(p^2 - 1)$? This question is closely related to an old problem in number theory which asks for the smallest prime in an arithmetic progression. It follows from [14] that $n(p) < \frac{1}{2}(p^2 - 1)$ for infinitely many primes p . In fact, it is a long standing conjecture in analytic number theory that if a and q are relatively prime and $\epsilon > 0$ then the smallest prime $p(q, a)$ congruent to a modulo q satisfies $p(q, a) = O(q^{1+\epsilon})$ (see [7, Chapter 18]). If true, this conjecture implies that the equality $f(p) = \frac{1}{2}p(p^2 - 1)$ can hold for only a finite number of primes p .

One can ask how often $n(p)$ is a prime power rather than a prime itself. Clearly, if p is a Mersenne prime, then $n(p) = p + 1$ is a power of 2. One can speculate that for any prime q there are infinitely many primes p such that $n(p)$ is a power of q . To prove or disprove any such claim seems beyond the reach of current methods. Perhaps more tractable is the question whether $n(p)$ is not a prime for infinitely many values of p . From the numerical evidence it seems that this happens rarely. For example, among the first 2000 primes only the Mersenne primes 3, 7, 31, 127, 8191, and the primes 13, 1093, 2801 yield $n(p)$ which is not a prime.

We end this discussion with the following curious characterization of Mersenne primes by group theoretic means.

Corollary 5.5. *For a prime p the following conditions are equivalent:*

- (i) $p = 2$ or a Mersenne prime;
- (ii) There is a non- p -closed group of order $p(p + 1)$.

Proof. Suppose that $p = 2$ or p is a Mersenne prime. Thus $n(p) = p + 1$. By Theorem 5.3 and Corollary 5.4, the non- p -closed group of smallest order has order $p(p + 1)$. Thus (ii) follows from (i).

Conversely, suppose there is a non- p -closed group of order $p(p + 1)$. Since $p = 3$ is a Mersenne prime, we may assume that $p > 3$. Thus $(p + 1) < (p^2 - 1)/2$ and we conclude by Theorem 5.3 that $n(p) \leq p + 1$. This is only possible if $n(p) = p + 1$ and p is a Mersenne prime. □

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