two millenia, Euclidean geometry was regarded as the pinnacle of deductive logic, until 19th century mathematicians realized that there were theorems that depended on implicit assumptions (such as the fact that the diagonals of a rhombus lie inside the figure) that were not logical deductions from the axioms.

It would be a mistake to assume that at last we have "got it right" and that this generation is free of the internal conflicts and confusions of the past. On the contrary, we have our own share of corporate creases of the mind. (See, for example, Sierpińska, 1985a, 1985b, 1987.) Many of the creases that we purport to see in students are actually present in ourselves and have been passed down in varyingly modified forms from generation to generation.

For example, the idea that a function $y = f(x)$ is single-valued has become part of our mathematical culture, and we may find it strange to see students asserting that a circle $x^2 + y^2 = 1$ can be a function. Yet the term "implicit function" continues to be used in textbooks to describe such an expression. I (to my eternal shame) find that I published a computer program called the "implicit function plotter" which will draw, among other things, the graph of $x^2 + y^2 = 1$. Likewise I find myself considering the draft of a new curriculum for the 16–19 age range in Britain which says of this equation: "Strictly speaking, y is not a function of x because there is not a unique value of y for each value of x, but we might think of it as a 'double-valued' function from x to y." What are students to think? Can any of us, with hand on our heart, state that we have never indulged in any vagaries of this kind? Let him who is without sin cast the first stone.

CONCEPT DEFINITION AND CONCEPT IMAGE

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. (Poincaré, 1914, p. 117)

The "new mathematics" of the 1960s was a valiant attempt to create an approach based on clear definitions of mathematical concepts, presented in a way (it was hoped) that students would understand. But it failed to achieve all its high ideals. The problem is that the individual's method of thinking about mathematical concepts depends on more than just the form of words used in a definition.

> Within mathematical activity, mathematical notions are not only used according to their formal definition, but also through mental representations which may differ for different people. These "individual models" are elaborated from "spontaneous models" (models which pre-exist, before the learning of the mathematical notion and which originate, for example, in daily experience) interfering with the mathematical definition. We notice that the notion of limit denotes very often a bound you cannot cross over, which can, or cannot, be approached. It is sometimes viewed as reachable, sometimes as unreachable. (Cornu, 1981)

Thus the experience of pupils prior to meeting formal definitions profoundly affects the way in which they form mental representations of those concepts. During the late 1970s and early 1980s many authors noted the mismatch between the concepts as formulated and conceived by formal mathematicians, and those as interpreted by students. For example, difficulties were noted in the understanding of the limiting process as secants tend to tangents (Orton, 1977), the meaning of infinite decimals (Tall, 1977), geometrical concepts (Vinner and Hershkowitz, 1980), the notion of function (Vinner, 1983), limits and continuity (Sierpińska, 1987; Tall & Vinner 1981), the meaning of the differential (Artigue, 1986), convergence of sequences (Robert, 1982), limits of functions (Ervynck, 1981), the tangent (Tall, 1987; Vinner, 1983), infinite series (Davis, 1982), infinite expressions (Borasi, 1985), the intuition of infinity (Fischbein, Tirosh, & Hess, 1979), and so on.

To highlight the role played by the individual's conceptual structure, the terms "concept image" and "concept definition" were introduced in Vinner and Hershkowitz (1980) and later described as follows:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.... As the concept image develops it need not be coherent at all times.... We will refer to the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. (Tall & Vinner, 1981, p. 152)

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On the other hand, "The *concept definition* [is] a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152).

The consideration of conflicts in thinking is widespre[ad in] the literature:

> New knowledge often contradicts the old, and effective learning requires strategies to deal with such conflict. Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be "kept around" if safely maintained in separate mental compartments. (Papert, 1980, p. 121)

> In general, learning a new idea does not obliterate an earlier idea. When faced with a question or task the student now has *two* ideas, and may retrieve the new one or may retrieve the old one. What is at stake is not the possession or non-possession of the new idea; but rather the *selection* (often unconscious) of which one to retrieve. Combinations of the two ideas are also possible, often with strikingly nonsensical results. (Davis & Vinner, 1986, p. 284)

This is particularly applicable to the transition to advanced mathematical thinking when the mind simultaneously has concept images based on earlier experiences that interact wit[h] new ideas based on definitions and deductions. The very ide[a] of *defining* a concept in a sentence, as opposed to *describing* it is at first very difficult to comprehend, particularly when ther[e] are words in the definition that are not defined. It is impossibl[e] to make a beginning without making some assumptions, an[d] these are based upon the individual's concept image, not o[n] any logically formulated concept definition.

* Better: The concept definition is the stipulated definition of the concept.