Solutions are in **boldface**. Most of the problems without solutions are routine computations.

All text problems are found on pages 112-115.

1. Composing permutations: Problem 17 (The author abbreviates $\beta \circ \alpha$ as $\beta \alpha$ and writes “product” instead of “composition.” That’s fine and it’s standard, but don’t forget that you are actually composing functions, not multiplying. The use of “product” is universal slang among group theorists, but it’s not necessarily good for novices. You should consider avoiding the slang.)

2. Cycle notation: Read the Cycle Notation Section on pages 97-99. By the way, the word “cycle” here should not be confused with the word “cyclic” as used in Chapter 3.

(a) Problem 18a. (By “product,” the author actually means “composition.” See above.)

(b) Write the permutations given in problems 4 and 17 as compositions of disjoint cycles.

(c) Now reverse the process by writing the permutations given in problem 3a,c,e,f in the array notation.

(d) Do problems 1, 2 (an educated guess based on problem 1 is all that I am after in problem 2); 3a,c with the definition only, without using any theorems.

(e) After doing part 2d above, read the statement of Theorem 5.3. Use that theorem to check your answers to 3a,c.

(f) Use Thm. 5.3 to do 3b,d,e,f and 20. Caution: 3e,f are not given initially as the composition of disjoint cycles.

(g) Problem 6

Consider the permutation $\alpha := (1; 2 3 4 5) \circ (6 7 8)$ in $S_8$. By Ruffini’s Theorem 5.3, $\alpha$ has order 15. By direct computation, $\alpha = (1 5) \circ (1 4) \circ (1 3) \circ (1 2) \circ (6 8) \circ (1 5)$, the composition of 6 2-cycles. Therefore, $\alpha$ is even, in other words, $\alpha \in A_8$.

(h) Problem 30

We did this one class.

3. A basic fact about $S_n$: Problem 41. Your task here is to take a generic $n$ which is at least 3 and show that $S_n$ is not Abelian.

Let $n$ be a positive integer with $n \geq 3$. Consider the permutations

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & n
\end{bmatrix} = (1 2 3) \quad \text{and} \quad
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & n
\end{bmatrix} = (1 2). \quad \text{Both are visibly in $S_n$. Moreover,} \quad
(1 2 3) \circ (1 2) = (1 3) \quad \text{while} \quad (1 2) \circ (1 2 3) = (2 3). \quad \text{Thus,} \quad
(1 2 3) \circ (1 2) \neq (1 2) \circ (1 2 3), \quad \text{showing that} \quad S_n \quad \text{is not abelian.}
$$

4. Writing permutations as compositions of 2-cycles: Problem 18b
5. Even and odd permutations: Problem 9

6. Thinking of symmetry groups as permutation groups: Problem 40

   Examples: Numbering the vertices of the triangle 1, 2, 3 in counterclockwise order, a counterclockwise rotation of 120 degrees may be represented by $(1 2 3)$, meaning vertex 1 goes to where vertex 2 used to be, vertex 2 goes to where vertex 3 used to be, and vertex 3 goes to where vertex 1 used to be. A counterclockwise rotation of 240 degrees may be represented by $(1 3 2)$. The others are similar.

7. Proofs about even and odd permutations: Problems 11 (provide a proof—it’s short), 12, 15. Important—notice in 11 the permutations are cycles, but in 12 and 15 they can be any permutations. These are particularly useful problems.

   Problem 11. Let $n$ be a positive integer. Take any $n$-cycle, $(a_1 a_2 \ldots a_n)$.

   Claim 1: If $n$ is an odd integer, then $(a_1 a_2 \ldots a_n)$ is an even permutation.

   Claim 2: If $n$ is an even integer, then $(a_1 a_2 \ldots a_n)$ is an odd permutation.

   By direct computation we know $(a_1 a_2 \ldots a_n) = (a_1 a_n) \circ (a_1 a_{n-1}) \circ \cdots \circ (a_1 a_2)$ which is the composition of $n - 1$ 2-cycles.

   Assume $n$ is an odd integer, then $n - 1$ is even. Therefore, $(a_1 a_2, \ldots a_n)$ is an even permutation by definition since it is the composition of an even number of 2-cycles. That proves the first claim.

   Assume $n$ is an even integer, then $n - 1$ is odd. Therefore, $(a_1 a_2, \ldots a_n)$ is an odd permutation by definition. That proves the second claim.

   Problem 12. Repeated warning, do not assume $\alpha$ and $\beta$ are cycles.

   Assume $\alpha$ is an even permutation. Then, by definition of an even permutation, $\alpha = (b_1 c_1) \circ (b_2 c_2) \circ \cdots \circ (b_k c_k)$ for some 2-cycles $(b_1 c_1), (b_2 c_2), \ldots, (b_k c_k)$ where $k$ is an even integer. Then $\alpha^{-1} = [(b_1 c_1) \circ (b_2 c_2) \circ \cdots \circ (b_k c_k)]^{-1}$

   $= (b_k c_k)^{-1} \circ (b_{k-1} c_{k-1})^{-1} \circ \cdots \circ (b_1 c_1)^{-1}$ (by Shoes and Socks)

   $= (c_k b_k) \circ (c_{k-1} b_{k-1}) \circ \cdots \circ (c_1 b_1)$ (by problem 30, done earlier)

   Therefore, since $k$ is an even integer, $\alpha^{-1}$ is an even permutation.

   (The odd case is similar.)

   Problem 15. Half was done in class. The other half is similar.

8. Short answer problems. In the interest of time, full proofs are optional. Write enough to explain your basic thinking, however.

   (a) Problem 14

   The first blank is completed this way: $r + s$ is even. The second blank: $r + s + t$ is odd. (Those are obtained by decomposing the cycles into the composition of 2-cycles in the usual way, and then counting the number of 2-cycles in terms of $r$, $s$, and $t$.)

   (b) Problems 24 and 25 (some discrete math!) (over $\leftrightarrow$)

   Problem 24 Count the number of elements of order 5 in $S_7$.

   Let’s first figure out what elements of order 5 look like in $S_7$. By Rufini’s
Theorem 5.3, the lcm of the lengths of the cycles in the disjoint cycle decomposition must be 5. Since 5 is prime, that means all the cycles have lengths 1 or 5. We only have the numbers 1, 2, 3, 4, 5, 6, 7 in the domain, so there are not enough elements to have more than 1 5-cycle. Thus, the elements of order 5 are just the 5-cycles (technically, composed with 2 1-cycles, but we don’t write those).

Now we use standard counting methods: There are 7 choices for the first entry in a 5-cycle, 6 choices for the second entry, 5 choices for the third, 4 for the fourth, and 3 for the fifth. Making a total of $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$. However, each 5-cycle is counted 5 times in that calculation since we can write it by starting at any of its entries. (Example: $(1 \ 2 \ 3 \ 4 \ 5) = (2 \ 3 \ 4 \ 5 \ 1) = \text{etc.}$)

Therefore, we have to divide by 5. Therefore, the number of elements of order 5 in $S_7$ is $(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)/5 = 504$.

9. (a) The set of all even permutations is a subgroup: Problem 13

   Proof: Recall the set of even permutations in $S_n$ is $A_n$, that is, $A_n := \{\beta \in S_n : \beta \text{ is even}\}$ (That was the definition stipulated in class). So we want to show $A_n \subseteq S_n$.

   (subset) $A_n \subseteq S_n$ because of the universal set of $A_n$.

   (closure) Closure follows immediately from problem 15.

   (identity) Since the identity is the composition of 0 2-cycles, it is even. Therefore, it is in $A_n$.

   (inverses) Problem 12 proved inverses.

(b) The set of all odd permutations a subgroup?: Problem 21

   No. Closure fails by problem 15, which tells us the composition of two odd permutations is even.

10. Prove Theorem 0.7 parts 2 & 3. Collaboration ban in effect for this problem.

   This one won’t be on the midterm. I’ll be reading the solutions that you submitted for homework.