

Final Exam Review

1. There are 3 slices of olive pizza, 5 slices of plain pizza, 7 slices of pepperoni pizza, and 8 slices of anchovy pizza remaining at a pizza party.

(a) How many slices need to be requested to assure that 3 of at least one kind of pizza are requested?

$2+2+2+2+1 = 9$ by Pigeonhole Principle

(b) How many slices need to be requested to assure that 3 slices of each kind of pizza are requested?

Worst case is if we got all of the anchovy, pepperoni and plain before we even selected ANY olives. Thus the answer is $8+7+5+3 = 23$ slices

2. At a party, there are 18 women and 10 men.

(a) How many ways are there to form 10 couples consisting of one man and one woman?

First we can pretend that the men are ordered alphabetically. (There is only one way to do that.) Then the first male selects one of the 18 women, the next male selects one of the 17 remaining women....the 10th male selects one of the remaining 9 women. Thus the answer is $18 \cdot 17 \cdot 16 \cdots 9 = P(18, 10)$. Another way to do it is to choose the 10 of the 18 women $\binom{18}{10}$. Then order the women $10!$ ways for that, so the answer is $\binom{18}{10}10! = P(18, 10)$.

(b) How many ways are there to form 8 couples consisting of one man and one woman?

Choose the 8 men (ordered alphabetically): $\binom{10}{8}$

Choose the 8 women: $\binom{18}{8}$. As in the previous part, you must order the women to go with the appropriate men. So the final answer is $\binom{10}{8}\binom{18}{8}8!$.

3. Among the graphs below, which pairs are isomorphic? If a pair is isomorphic, then label the vertices appropriately.

4. Find the probability of being dealt a 7-card poker hand that consists of one three-of-a-kind and two pairs. (3 cards of one rank, and 2 different sets of 2 matching ranks)

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2}{\binom{52}{7}}$$

5. How many integral solutions of $x_1 + x_2 + x_3 + x_4 = 27$ satisfy $x_1 \geq -4$, $x_2 \geq 3$, $x_3 \geq 1$, and $x_4 \geq 12$.

We make $x'_1 = x_1 + 4$, $x'_2 = x_2 - 3$, $x'_3 = x_3 - 1$, $x'_4 = x_4 - 12$ to get $x'_1 + x'_2 + x'_3 + x'_4 = 15$. The number of non-negative integral solutions to this is $\binom{15+4-1}{4-1} = \binom{18}{3}$.

6. Use the fact that $\binom{n}{k} = \binom{n}{n-k}$ to prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Note that $\sum_{k=0}^n \binom{n}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k}$. Pretend we have a $(2n)$ -set S and we divided S into two n -sets, A and B , and we want to choose a total of n elements from S . Then, we can either choose $0, 1, 2, \dots, n$ elements from A . When we choose k elements from A , then the remaining $n - k$ elements must come from B , since we want to choose a total of n elements from S . This is true for all k . Note that there are $\binom{n}{k}$ ways to choose k elements from A and for each of those, there are $\binom{n}{n-k}$ ways to choose the remaining $n - k$ elements from B . Also, there are $\binom{2n}{n}$ ways to choose n elements from S .

We therefore have that $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2$.

7. There is an even number of $2n$ people at a party, and they talk together in pairs with everyone talking with someone (so n pairs). In how many different ways can the $2n$ people be talking like this?

The first person in alphabetical order, say, chooses the person they wish to talk to: $2n - 1$ ways to do that. This leaves $2n - 2$ people left. The next person in alphabetical order, not used yet, chooses the person, among the remaining $2n - 3$ people, they wish to talk to. This keeps going, so the answer is

$$\begin{aligned} (2n - 1)(2n - 3) \cdots 5 \cdot 3 \cdot 1 &= (2n - 1)(2n - 3) \cdots 5 \cdot 3 \cdot 1 \frac{(2n)(2n - 2)(2n - 4) \cdots 4 \cdot 2}{(2n)(2n - 2)(2n - 4) \cdots 4 \cdot 2} \\ &= \frac{(2n)!}{2^n n!} \end{aligned}$$

8. Now suppose that there is an odd number $2n + 1$ of people at the party with everyone but one person talking with someone. How many different pairings are there?

All we do differently from #7 is select the odd person out. There are $2n + 1$ ways to select that odd person out. Thus, there are

$$\frac{(2n + 1)!}{2^n n!}$$

ways to pair off these people.

9. At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that

- (a) no gentleman receives his own hat?

$$D_7$$

- (b) at least two of the gentlemen receive their own hat?

Get rid of all the permutations where either zero or one person received the correct hat. Thus the answer is $7! - D_7 - 7D_6$.

10. Find the sum of all positive integers less than 6000 which are multiples of 3, but are not multiples of 5.

$$(3 + 6 + 9 + \cdots 5997) - (15 + 30 + 45 + \cdots 5985) = \frac{6000 \cdot 1999}{2} - \frac{6000 \cdot 399}{2} = 4800000$$

11. Recall the Fibonacci sequence, where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. Be evaluating the following expression for small values of n , conjecture a general formula and then prove it, using induction.

$$f_0 + f_2 + f_4 + \cdots + f_{2n}$$

$$f_0 = 0$$

$$f_0 + f_2 = 0 + 1 = 1 = f_3 - 1$$

$$f_0 + f_2 + f_4 = 0 + 1 + 3 = 4 = f_5 - 1$$

$$f_0 + f_2 + f_4 + f_6 = 12 = f_7 - 1$$

$$\text{Therefore } f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1.$$

Proof by induction: Base case. Check. Assume true for $k < n$, (i.e. $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1$) and prove for n .

WTS : $f_0 + f_2 + \cdots + f_{2n} + f_{2n+2} = f_{2n+3} - 1$. By induction hypothesis, $LHS = (f_{2n+1} - 1) + f_{2n+2} = f_{2n+3} - 1 = RHS$ by the recurrence relation for the Fibonacci numbers.

12. (a) Count the cycles of length n in K_n .

$\frac{(n-1)!}{2}$. You did this already.

- (b) Count the cycles of length $2n$ in $K_{n,n}$.

Answer:

$$\frac{n!(n-1)!}{2}$$

Try to figure out why this is true.