MTH 355 Discrete Mathematics

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What is "Discrete Mathematics"?

You will find many definitions of discrete mathematics out there. For the purposes of this class, we will use the following: “Discrete mathematics is used whenever objects are counted, when relationships between finite sets are studied, and when processes involving a finite number of steps are analyzed.” - Kenneth Rosen. For me, discrete mathematics is about counting, playing games, coloring, and solving puzzles.

Discrete mathematics problems can be classified into three broad categories. The first category, existence problems, deals with whether or not a given problem has a solution. The second category, counting problems, investigates how many solutions may exist for problems with known solutions. A third category, optimization problems, focuses on finding a best solution to a particular problem. We will primarily focus on the first two category of problems. We will mostly study combinatorics, which, according to your book, is “concerned with arrangements of the objects of a set into patterns satisfying specified rules.” You have probably done combinatorics in probability class or other times you may have never thought of. Basic combinatorics problems involve counting. For example, if Baskin Robbins has 31 flavors of ice cream, and you send your 10 kids in to buy one scoop ice cream cones, how many different combination of cones are possible? When we ask this, we might also ask ourselves
(i) if the order of how the cones are selected matters (no),
(ii) if multiple people can order the same flavor of ice cream cone (yes),
(iii) if I order a cone (we'll say no)...

I cannot emphasize the following statements by the author enough: “The solutions of combinatorial problems can often be obtained using ad hoc arguments, possibly coupled with use of general theory. **One cannot always fall back on application of formulas or known results**.” To me, this means you have to play around sometimes. The answer is not always right in front of your face. I got into mathematics because I love solving puzzles. Discrete mathematics is that for me.

To solve a combinatorial problem, you might have to perform the following steps:

1. Make sure you understand what the question is asking.
2. Determine what AN example of an object satisfying the requirements of the problem might look like.
3. Do some computations for small cases in order to develop an idea of what exactly is going on. (This is probably most important)
4. Try to develop some sort of systematic approach when doing small examples.
5. Use reasoning and possibly creativity to obtain a solution to the problem.

Chapter 1

Let's begin with some basic problems, which are actually kind of fun.

Perfect Covers of Chessboards: An ordinary chessboard is divided into 64 squares (8 in each row and column). The squares on the board alternate between black and white, where adjacent squares are of different colors. Suppose a domino fits perfectly over two adjacent squares.

Question 1: Can you "perfectly" cover a chessboard with dominoes?

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Question 2: Suppose you remove the top right and bottom left squares of the chessboard. Can you cover the modified chessboard with dominoes?

Question 3: Suppose you remove the top right and bottom right squares of the original chessboard. Can you perfectly cover this modified chessboard with dominoes?

Question 4: For what values of n can you perfectly cover an $n \times n$ chessboard with dominoes?

A domino can be thought of as a 1×2 piece. A b-omino is a $1 \times b$ piece. Your book mentions the following theorem:

Theorem (b-omino coverings of $m \times n$ chessboards): An $m \times n$ chessboard has a perfect covering by b-ominoes if and only if b is a factor of m or b is a factor of n. (If there's time, we will prove this.)

The (Secret of the) Game of Nim

Nim is a game played by two players, where there are k heaps of stones of sizes $n_1, n_2, \ldots, n_k$. The players alternate turns, and each player during his/her turn follows this simple rule:
Select exactly one of the heaps and remove at least one stone (and possibly all the stones) from the selected heap.
The game ends when all the heaps are empty. The player to select the last stone is the winner of the game.

The variables in this game are the number of heaps (k) and the size of each of the heaps (the n_i's). The problem is determining who wins the game and what is the winning strategy. Let's start with a few simple exams.

Question 1: If there is one heap, who wins the game? What is the winning strategy?

Now play the following games with your neighbor. Each of these games has 2 heaps. In each case, determine the winner and a winning strategy.

Game 1: $n_1 = 3, n_2 = 3$

Game 2: $n_1 = 3, n_2 = 1$

Game 3: $n_1 = 5, n_2 = 2$

Question 2: If there are two heaps, who wins the game?

More than two heaps: It gets a little harder for more than two heaps, but the general principle is still there.

Consider the Nim game with heaps of size $n_1, n_2, \ldots, n_k$. Express each n_i in base 2, Thus

$$n_1 = a_s a_{s-1} \cdots a_1 a_0$$

$$n_2 = b_s b_{s-1} \cdots b_1 b_0$$

$$\cdots$$

$$n_k = g_s g_{s-1} \cdots g_1 g_0,$$

where $a_m = 0$ or 1 and represents the coefficient of 2^m in the binary expansion of n_1.

We say the game is **balanced** if $\forall 1 \leq j \leq s$, we have $a_j + b_j + \cdots + g_j$ is even, i.e.,

$$a_0 + b_0 + \cdots + g_0 \text{ is even,}$$

$$a_1 + b_1 + \cdots + g_1 \text{ is even,}$$

$$\cdots$$

$$a_s + b_s + \cdots + g_s \text{ is even,}$$

A game is unbalanced otherwise.

Theorem:

Example 1: Consider the 3-heap Nim game with $n_1 = 5, n_2 = 7, n_3 = 6$. Is the game balanced or unbalanced? Who wins and how?

Example 2: Consider the 4-heap Nim game with $n_1 = 20, n_2 = 10, n_3 = 19, n_4 = 13$. Is the game balanced or unbalanced? Who wins and how?

Induction Review

Induction: Assume you want to prove that for some statement P, P(n) is true for all n starting with $n = 1$. The **Principle (or Axiom) of Math Induction** states that, to this end, one should accomplish just three steps:

1. Base Case: Prove that P(1) is true.
2. Induction Hypothesis: Assume that P(n) is true for some n.
3. Induction Step: Derive from here that P($n + 1$) is also true.

The idea of Mathematical Induction is that a finite number of steps may be needed to prove an infinite number of statements P(1), P(2), P(3), ...

Example 1: Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for any natural number n.

Example 2: Determine a closed form formula for the following sum ($n \geq 1$):

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

Prove that your formula is correct using induction.

Intuitively, the inductive (third) step allows one to say, look P(1) is true and implies P(2). Therefore P(2) is true. But P(2) implies P(3). Therefore P(3) is true which implies P(4) and so on. Mathematical induction is just a shortcut that collapses an infinite number of such steps into the three above.

Strong Induction: In this case, we again need three steps:

1. Base Case: Prove that P(1) is true.
2. Induction Hypothesis: Assume that P(k) is true for all $k < n$ for some n.
3. Induction Step: Derive from here that P(n) is also true.

Example 3: Use strong induction to show that for any positive integer $n > 1$, the sum

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Example 4: Prove that $11^n - 6$ is divisible by 5 for any positive integer n.

§2.1 **Basic Counting Rules**

Example 1: Suppose there are three major routes from Washington DC to Chicago, and four from Chicago to Los Angeles. How many routes are there from DC to LA that go through Chicago?

Example 2: Suppose you roll a regular 6-sided die and a 4-sided die simultaneously. How many different combinations can you have? (What does this mean exactly?)

Multiplication Rule If something can happen in n_1 ways, and no matter how the first thing happens, a second thing can happen in n_2 ways, then the two things together can happen in $n_1 \cdot n_2$ ways.

More generally, if something can happen n_1 ways, and no matter how the first thing happens, a second thing can happen in n_2 ways, and no matter how the first two happen, a third thing can happen in n_3 ways, and ..., then all the things together can happen in $n_1 \cdot n_2 \cdot n_3 \cdots$ ways.

Example 3: The population of Carlisle, Pennsylvania, is about 20,000. If each resident has three initials, is it true that there must be at least two residents with the same initials?

Example 4: How many non-negative integers are less than 1000 and contain the digit 2?

Example 5: There are 100 senators and 435 members of the House of Representatives. A delegation is being selected to see the President. In how many ways can such a delegation be picked if it consists of one senator *and* one representative?

What if the delegation is to consist of one member of the Senate *or* one member of the House?

Sum Rule: If one event can occur n_1 ways, a second event can occur in n_2 (different) ways, a third event can occur in n_3 (still different) ways, ..., then there are $n_1 + n_2 + n_3 + \cdots$ ways in which (exactly) one of the events can occur.

Example 6: A committee is to be chosen from 8 scientists, 7 psychics, and 12 clerics. If the committee is to have two members of different backgrounds, how many committees are there?

§2.2 **Permutations**

Example 1: How many different words can be formed by rearranging the letters of the word MATH?

We may refer to a set consisting of n distinct elements as an n-set. A **permutation** of an n-set is an arrangement of the elements of the set in order. (Therefore rearranging the letters of MATH can be thought of as permutations of a 4-set.)

Example 2: List the permutations of $\{1, 2, 3\}$.

How many permutations are there of an n-set?

Example 3: How many permutations of $\{1, 2, 3, 4, 5\}$ begin with 5?

Example 4: How many different ways can 10 people sit around a circular table? (Since the table is circular, there is no "head" of the table, i.e., there is no position #1.)

Example 5: A veterinarian has 4 patients which are cats, 4 which are dogs and 4 which are birds. She has 4 morning, 4 afternoon and 4 evening appointments slots. In how many ways can she schedule appointments for the animals if she only wants to see one type of animal per time of day (i.e. all cats could be seen in the afternoon, all dogs in the evening and all birds in the afternoon)?

r-Permutations

Given an n-set, suppose that we want to pick out r elements and arrange them in order. Such an arrangement is called an **r-permutation of the n-set**. The number of such permutations is denoted ${}_nP_r$ or $P(n, r)$.

Example 6: Let $A = \{a, b, c, d, e, f, g, h\}$. Find the number of sequences of length 3 using elements of A if no element of A is to be used twice.

Calculate $P(n, r)$ (where $n \geq r \geq 0$).

Example 7: If a campus telephone extension has four digits, how many different extensions are there with no repeated digits:

(a) If the first digit cannot be 0?

(b) If the first digit cannot be 0 and the second cannot be 1?

Example 8: How many permutations are there of the letters of the word REFER?

§2.3 **Subsets and Combinations**

Subsets

Example 9: Naples pizza has 7 toppings available: {Anchovies, Bacon, Extra Cheese, Mushroom, Onion, Pepperoni, Sausage}. They advertise that they offer over 100 varieties of pizza. At the pizza place, it is possible to have on a pizza a choice of any combination of the 7 toppings. Is the pizza place telling the truth?

Note that Example 9 asks for the number of subsets of a 7-set.

How many subsets are there of an n-set?

Example 10: If Naples decides to always put onions and mushrooms on its pizzas, how many different varieties can the shop now offer?

Example 11: If A is a set of 8 elements, how many subsets of more than one element does A have?

Combinations

Definition: An k**-combination of an** n**-set** is a selection of k elements from the set, where order does NOT matter.

Example 1: If you have 5 toppings available for pizza {Bacon, Mushroom, Onion, Pepperoni, Sausage}, how many 3-topping pizzas can we make?

The number of k-combinations of an n-set is denoted $C(n, k)$ or ${}_nC_k$ or $\binom{n}{k.}$ We will usually use the last notation $\binom{n}{k}$ and say "n choose k."

$$\binom{n}{k} =$$

Proof:

Example 2: If there are 6 drugs being tested in an experiment, and we want to choose 3 of them to give to a particular subject, then how many ways can this be done?

Example 3: How many 8-letter words can be constructed by using the 26 letters of the alphabet if each word contains 3,4 or 5 vowels? It is understood that there is no restriction on the number of times a letter can be used in a word.

Example 4: At a party, there are 15 men and 20 women.

(a) How many ways are there to form 15 couples consisting of one man and one woman?

(b) How many ways are to form 10 couples consisting of one man and one woman?

Example 5: A fleet is to be chosen from a set of 8 different make foreign cars and 5 different make domestic cars. How many ways are there to form the fleet if:

(a) The fleet has 6 cars, 4 foreign and 2 domestic?

(b) The fleet can be any size (except empty), but it must have equal numbers of foreign and domestic cars?

(c) The fleet has 6 cars, 3 of each kind, and a Ford and BMW cannot both be in the fleet?

Claim 1:

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof of Claim 1:

Claim 2:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof of Claim 2:

Combinations and Pascal's Triangle

§2.4 **Permutations of Multisets**

So far we have mainly been selecting one subset of objects. We may sometimes wish to place all of our objects and determine where they are.

Example of an r-permutation of an n-set ($n = 82, r = 3$): In the National Hockey League (NHL), a team can either win, lose, or tie each of its games. In an 82-game schedule, how many different seasons can a particular team have? (A "season" is not just the teams records, but how it was obtained, i.e. order matters. For example, a team that only won its first game had a different season than a team that only won its last game.)

In the above example, we are sampling with replacement. We are choosing an 82-permutation of a 3-set, with replacement (i.e. we are allowing repetition). Let $P^R(n, r)$ be the number of **r-permutations** of an n-set, with repetition allowed (where **order matters**). Then

$$P^R(n, r) =$$

Now we count permutations of a multiset with objects of k different types. Suppose we have n_1 objects of the first type, n_2 objects of the second type, $\ldots$, n_k objects of the k^{th} type. Let $\binom{n}{n_1, n_2, \ldots, n_k}$ be the number of permutations of these objects, where $n = n_1 + n_2 + \cdots + n_k$. We will determine a formula for this.

Example 1 The university's registrar's office is having a problem. It has 13 new students to squeeze into 4 sections of an introductory course: 3 in the first, 4 each in the second and third, and 2 in the fourth. In how many ways can this be done?

Now let us derive a formula for

$$\binom{n}{n_1, n_2, \ldots, n_k} =$$

Example 2: A code is being written using the five symbols $+, \sharp, \bowtie, \nabla, \otimes$.

a How many 10-digit codewords are possible?

b How many 10-digit codewords are there that use exactly 2 of each symbol?

Example 3: How many ways are there to arrange the letters of the word *Mississippi*?

Example 4: How many ways are there to form a sequence of 10 letters from $4a$'s, $4b$'s, $4c$'s and $4d$'s if each letter must appear at least twice?

§2.5 **Combinations of Multisets**

We can also talk about k-**combinations** of an n-set with repetition allowed (where **order does not** matter). The number of k-combinations of an n-set with repetition allowed will be denoted $C^R(n,k)$.

Example 2: Find all 2-combinations of a 3-set.

Theorem:

$$C^R(n,k) = \binom{n+k-1}{n-1}$$

Proof:

Example 3: Baskin Robbins has 31 flavors of ice cream. If 10 kids go to Baskin Robbins for a single scoop of ice cream, how many different combinations of cones can they get?

Example: Integer solutions to linear equations:

(i) How many sequences of non-negative integers $x_1, x_2, \ldots, x_k$ are there that satisfy the equation $x_1 + x_2 + \ldots x_k = r$, where r is a positive integer?

(ii) How many sequences of **positive** integers $x_1, x_2, \ldots, x_k$ are there that satisfy the same equation?

Choosing a Sample of k Elements from a Set of n elements

Order counts?	Repetition allowed?	The sample is called	Number of ways to choose the sample
No	No	k-combinations	
Yes	No	k-permutations	
No	Yes	k-combinations with replacement	
Yes	Yes	k-permutations with replacement	

§2.6 **Probability**

It should be noted that the history of discrete mathematics is closely intertwined with the history of the theory of probability. Recall the definition of the probability of an event occurring.
Definition: The **probability** of an event is the number of "good" outcomes divided by the total number of possible outcomes.

Example 1: Calculate the probability that when a die is tossed, the outcome will be a number divisible by 3.

To make things precise, we say that an **experiment** is conducted that produces one of a number of possible outcomes. The set of possible outcomes is called the **sample space**. A subset of these outcomes is called an **event**. We'll write $P(A)$ to denote the probability of A occurring. If we let A be an event, then the **complement** of A, denoted A', is those outcomes that signal that the event has *not* taken place. Therefore $P(A') = 1 - P(A)$.

Example 2: Calculate the probability that a family of 3 children has

(i) exactly 2 boys (ii) At least one boy and at least one girl

Recall that $A \cup B$ is the event that indicates the **union** of events A and B, i.e., the event that either A **or** B occurred or the event that they both occurred. In Venn Diagram form, it looks like:

Recall that $A \cap B$ is the event that indicates the **intersection** of events A and B, i.e., the event that both A **and** B occurred. In Venn Diagram form, it looks like:

It should be noted that

$P(A \cup B) =$

Example 3: Suppose you toss a fair coin 6 times. Calculate the probability of getting each of the following:

(i) Two heads or Two tails;

(ii) Two heads or a head on the first toss;

(iii) An even number of heads and a head on the first toss.

Example 4: A bagel store sells six different kinds of bagels. Suppose you choose 15 bagels at random.

(i) What is the probability that your choice contains at least one bagel of each kind?

(ii) If one of the kinds of bagels is Sesame, what is the probability that your choice contains at least three Sesame bagels?

§3.1 **The Pigeonhole Principle**

Example of Pigeonhole Principle: How many people do you have to have in a room in order to guarantee that two people have the same first initial? ____________

Pigeonhole Principle: If $n+1$ objects are placed into n boxes, then there is at least one box with two or more objects.

Example 1: Suppose my CD collection consists of 5 albums by Kelly Clarkson and 11 albums by Incubus

(a) How many CDs must I select in order to get two by the same artist?

(b) How many CDs must I select in order to get two by each artist?

Example 2: Given m integers $a_1, a_2, \ldots, a_m$, there exist integers k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \cdots + a_l$ is divisible by m.

Example 3: From the integers $1, 2, \ldots, 100$, we choose 51 integers. Show that among the integers chosen, there are two such that one of them is divisible by the other.

Generalized Pigeonhole Principle: Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are distributed into n boxes, then either the first box contains at least q_1 objects, the second box contains at least q_2 objects, ..., or the n^{th} box contains at least q_n objects.
Proof:

Example 4: How many people must be in a room to guarantee that at least three people will have a common birth month?

Example 5: We have 10 boxes labeled 1 through 10 into which we place pennies. How many pennies are required to ensure that at least one box contains at least as many pennies as the label on the box?

Before we do another example, let's look at some definitions. Consider the sequence of numbers $x_1, x_2, \ldots, x_p$. A **subsequence** is any sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_q}$ such that $1 \leq i_1 < i_2 < \cdots < i_q \leq p$. For instance if $x_1 = 9, x_2 = 6, x_3 = 14, x_4 = 8$, and $x_5 = 17$, then a subsequence x_2, x_4, x_5 is the sequence 6, 8, 17. A subsequence is **increasing** if its entries go successively up in value (so the subsequence listed above is increasing), and **decreasing** if its entries go successively down in value. The subsequence listed is also the *longest* increasing subsequence.

Example 6 (Erdős-Szekeres Theorem): Given a sequence of $n^2 + 1$ distinct integers, either there is an increasing subsequence of $n + 1$ terms or a decreasing subsequence of $n + 1$ terms.

§4.5 **Equivalence Relations and Partial Orders**

Given a set of objects, sometimes we would like to order the objects in the set. There are times, though, that is not possible to order the elements. For example, if you were to order playing cards. When we talk about order, we need a relationship between the elements of the set, so we can determine how to order them. Such relations are $\leq, \subseteq$, or divisibility.

We can develop **partial orders** for some of these sets, which are unable to be **totally ordered**. We must now give a precise definition of partial order. Let X be a set and $a, b \in X$. The relation R is a subset of $X \times X$, the ordered pairs of elements of X. We say aRb ("a is related to b") provided $(a, b) \in R$.

Here is a list of some properties that a relation R on a set X can have:

1. R is **reflexive** if $xRx, \forall x \in X$.
2. R is **irreflexive** if $x \not R x, \forall x \in X$.
3. R is **symmetric** if xRy if and only if yRx for all $x, y \in X$.
4. R is **antisymmetric** if xRy implies $y \not R x$ for all $x \neq y \in X$.
5. R is **transitive** if xRy, yRz implies xRz for all $x, y, z \in X$.

Let's look at the relations R discussed earlier ($\leq, \subseteq$, and divisibility) to see which of these properties are satisfied by each relation using $X = \mathbb{N}$.

A relation R is a **partial order** on a set X if it is:

(i) reflexive, (ii) antisymmetric, and (iii) transitive.

Question: Of the previous relations discussed, were any of them partial orders on $\mathbb{N}$?

A relation R is an **equivalence relation** on a set X if it is:

(i) reflexive, (ii) symmetric, and (iii) transitive.

We usually write, $a \sim b$ to represent an equivalence relation. Equivalence classes partition a set!

Question: Of the previous relations discussed, were any of them equivalence relations on $\mathbb{N}$? What about your relatives? Is familial relations an equivalence relation?

One can think of an equivalence relation as "=" and a partial order as $\leq$.

Let's answer a couple basic questions involving equivalence relations and partial orders.

Example 1: (§4.6#44) Let $A_1, A_2, \ldots, A_s$ be a partition of a set X. Define a relation R on X by xRy if and only if x and y belong to the same part of the partition. Prove that R is an equivalence relation.

Example 2: (§4.6#48) Consider the partial order $\leq$ on the set X of positive integers given by "is a divisor of." Let a and b be two integers. Let c be the largest integers such that $c \leq a$ and $c \leq b$. Similarly, let d be the smallest integer such that $a \leq d$ and $b \leq d$. What are c and d?

Example 3: (§4.6#49) Prove that the intersection $R \cap S$ of two equivalence relations R and S on a set X is also an equivalence relation on X. What about $R \cup S$?

§5.2 **Binomial Expansion**

Theorem 1 (Binomial Expansion) For $n \geq 0$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof:

Example 1: Find the coefficient of x^{11} in the expansion of $(2+x)^{14}$.

Example 2: Find the coefficient of x^6 in the expansion of $(1+x)^8(1+2x)^4$.

Theorem 2 For $n \geq 0$,

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \underline{\hspace{4cm}}.$$

Proof(s): **Theorem 3** For $n > 0$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = \underline{\hspace{4cm}}.$$

Corollary:

Example 3: Find $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$ $\left(\text{You can use a similar argument to find } \displaystyle\sum_{k=0}^{n} k^2\binom{n}{k}\right)$

Example 4: Show $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

§5.4 **The Multinomial Theorem**

Recall from section §2.4 that $\binom{n}{n_1, n_2, \ldots, n_k} = \dfrac{n!}{n_1!n_2!\cdots n_k!}$, where $n_1 + n_2 + \cdots + n_k = n$. We call this number a **multinomial coefficient**.

The Multinomial Theorem: Let n be a positive integer. For all $x_1, x_2, \ldots, x_k$,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where we sum over all integral solutions of $n_1 + n_2 + \cdots + n_k = n$.

Example 1: Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$.

Example 2: What is the coefficient of a^3bc^2 in the expansion of $(4a - 2b + 5c)^6$?

§6.1 **The Inclusion-Exclusion Principle**

Example 1: Suppose that in a group of 18 job applicants, 10 have computer programming expertise, 5 have statistical expertise, and 2 have both programming and statistical expertise. How many of the group have neither expertise?

Example 2: How many numbers between 1 and 60 (inclusive) are not divisible by 2, 3 or 5? How many number between 1 and 750 (inclusive) are not divisible by 2, 3 or 5?

Theorem The Principle of Inclusion-Exclusion: Suppose we have a set of objects S and a set of properties $P_1, P_2, \dots, P_m$. For $i = 1, \dots m$, let A_i be the elements in S with property i.

The number of objects S that have none of the properties $P_1, P_2, \dots, P_m$ is given by

$$|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m| =$$

Corollary: The number of objects in S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by

$$|A_1 \cup A_2 \cup \cdots A_m| =$$

Example 3: How many permutations of the letters M, A, T, H, I, S, F, U, N are there such that none of the words MATH, IS, and FUN occur as consecutive letters? (Thus for instance, the permutation MATHISFUN is not allowed, nor are the permutations INUMATHSF and ISMATHFUN.)

Example 4: How many integers between 0 and 99,999 (inclusive) have among their digits each of 2, 5 and 8?

§6.3 **Derangements**

Example (Cellphone check problem): Imagine that n math majors attend class and check their cell phones at the door because their professor does not like when they use them in class. The professor is a little absent-minded (as usual) and returns the cell phones at the end of class randomly. What is the probability that no student receives his/her own cell phone back? Determine this answer for n = 2, 3, 4, 5. We'll solve this multiple ways.

Method 1 (Inclusion-Exclusion Principle):

Method 2 (Recurrence Relations):

Example of Derangements: Suppose you have eight young wizards who turn in their wands into Severus Snape for misbehaving. These wizards are Albus, Cho, Draco, Fred, George, Hermione, Neville, Ron. Because Harry has given Snape an amnesia charm, Snape randomly hands back the wands to the students and no wizard gets their own wand back.

1. Suppose that Albus', Cho's, Draco's and Fred's wands have been handed back to those same four wizards (but still no one gets their own wand), how many different ways is there to distribute all EIGHT wands back to the wizards if this happened?

2. Suppose that Albus', Cho's Draco's and Fred's wands have been handed back to the other four wizards (but still no one gets their own wand), how many different ways is there to distribute all SIX wands back to the wizards if this happened?

§7.1 **Recurrence Relations**

We frequently want to count a quantity a_k which depends on a parameter k. Some simple examples are:

- The number of permutations of n objects, $p_n = n!$.
- The number of proper subsets of an n-set is 2^{n-1}.
- $h_n = 2n - 1$ for $n \in \mathbb{N}$.

Sometimes, these values depend on the previous values. For example,

- $h_n = 3h_{n-1} + h_{n-2}, n \geq 3$
- $D_n = nD_{n-1} + (-1)^n, n \geq 2$.

Below is an example of such an occurrence, for which we would like to find the actual recurrence.

Example 1: Codewords from the alphabet $\{0, 1, 2, 3\}$ are to be recognized as *legitimate* if and only if they have an even number of 0's. How many legitimate codewords are there of length k? (Let a_k be the answer. We will try to find a relationship between a_{k+1} and a_k for $k \geq 0$.)

Example 2: A line separates the plane into two regions. Two intersecting lines separate the plane into four regions. Suppose that we have n lines in "general position"; that is no two are parallel and no three lines intersect at the same point. Into how many regions do these lines divide the plane?

Example 3: The Grand Daddy of all Recurrences: In 1202, Leonardo of Pisa (a.k.a. Fibonacci) asked the following question about rabbits. Suppose we start with one pair of adult rabbits (of opposite gender). Assume that each pair of rabbits produce one pair of young (of opposite gender) each month. A newborn pair of rabbits become adults in two months, at which time they also produce their first pair of young. Assume that rabbits never die and we are not concerned with in-breeding. Let f_k count the number of rabbit pairs present at the beginning of the k^{th} month. Derive a recurrence for f_k.

Alternative ways to think about the Fibonacci sequence: Suppose we have 1 x 1 squares and 1 x 2 dominoes. Let F_n be the number of ways to tile a 1 x n board using squares and dominoes. Find a recurrence relation for F_n. Also, find a relationship between F_n and f_n.

Example 4: Prove that for $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

Example 5: Prove that for $n \geq 0$, $\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = f_n$. (We'll actually prove $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_n$.)

7.2 Generating Functions

Recall that a power series of a function $f(x)$ (which has a derivative for all orders of x in an interval containing 0) looks like:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Such a series above is also called a *Taylor series expansion* for f about $x = 0$ or a *Maclaurin expansion* for f. Below are a few examples of such series

$\frac{1}{1-x} =$

$e^x =$

$\ln(1+x) =$

Example 1: Write the power series for $\ln(1+x^4)$.

Note that if a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for all $|x| < R$ with $R > 0$, the derivative and antiderivative of $f(x)$ can be computed by differentiating and integrating term by term. Namely

$f'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) =$

$\int_0^x f(t)dt =$

Example 2: Find the power series representation of $\frac{1}{(1-x)^2}$.

Suppose that we are interested in computing the k^{th} term in a sequence $\{a_k\}$ of numbers. We shall use the convention that $\{a_k\}$ refers to the sequence and a_k to the k^{th} entry. The **(ordinary) generating function** for the sequence $\{a_k\}$ is defined to be

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_o x^0 + a_1 x^1 + a_2 x^2 + \cdots$$

Example 3: Suppose that $a_k = \binom{n}{k}$, for $k = 0, 1, \ldots, n$. Then the ordinary generating function for the sequence $\{a_k\}$ is

$G(x) =$

Example 4: The functions below are each the ordinary generating function for a sequence $\{a_k\}$. Find the sequence corresponding to each function.

$$\left(\frac{5}{1-x}\right)\left(\frac{3}{1-x}\right)$$

$$\frac{x^3 + x^5}{1-x}$$

Example 5: Find a simple, closed-form expression for the ordinary generating function of the following sequence $\{a_k\}$. $a_k = 3k + 4$.

Counting Problems involving Generating Functions

Example 6 (Like §7.2 pg. 219): Determine the number h_n bags of fruit that can be made out of apples, bananas, oranges, and pears, where, in each bag, the number of apples is odd, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

Example 7: Determine the number of nonnegative solutions of $2x_1 + 5x_2 + x_3 + 7x_4 = n$.

Example 8 (Sicherman Dice): Suppose that a standard pair of dice are rolled. What are the probabilities for the various outcomes for this roll?

Does there exist a different pair of six-sided dice (that are not numbered 1 through 6) which yield the same outcome probabilities as standard dice? Let $a_1, a_2, a_3, a_4, a_5, a_6$ be the values on one die and $b_1, b_2, b_3, b_4, b_5, b_6$ be the values on the other die.

§7.3 **Exponential Generating Functions**

We had previously covered **ordinary** generating functions. We used these generating functions when order did not matter, i.e., we were talking about combinations. When order does matter, we use **exponential generating functions**. So for a sequence $\{a_k\}$, its exponential generating function is $H(x) = \sum_{k=0\infty} a_k \frac{x^k}{k!}$.

Why do we need **exponential** generating functions?

Recall that $P(n,k)$ is the number of k-permutations of an n-set. Thus the ordinary generating function for $P(n,k)$ with n fixed is given by:

$$G(x) = P(n,0)x^0 + P(n,1)x^1 + \cdots + P(n,n)x^n$$

Example 1: A code can use three different letters, a, b, or c. A sequence of five or fewer letters gives a codeword. The codeword can use at most one b, at most one c, and at most three a's. How many possible codewords (that only use those three letters) are there of length k for $k = 0, 1, 2, 3, 4, 5.$?

Theorem: Suppose that we have p types of objects, with n_i indistinguishable objects of type i, $i = 1, 2, \ldots, p$. The number of distinguishable permutations of length k with up to n_i objects of type i is the coefficient of $x^k/k!$ in the exponential generating function

$$\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right)\cdots\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_p}}{n_p!}\right)$$

Example 2: For each of the following questions, set up the appropriate generating function and indicate what you are looking for. For example, "the coefficient of x^8 in $\dfrac{2x+5}{(1-x^2)(1+x^4)}$."

a. A codeword consists of at least one of each of the digits 0, 1, 2, 3, and 4, and has length 6. How many such codewords are there?

b. In how many ways can $3n$ letters be selected from $2n$ A's, $2n$ B's, and $2n$ C's?

Example 3 (§7.3 pg. 226): Determine the number of ways to color the square of a 1-by-n chessboard, using the colors, red, white, and blue, if an even number of squares are to be colored red.

§7.4 **Solving Linear Homogeneous Recurrence Relations**

We have derived recurrence relations, but not really solved them yet. Consider the recurrence

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots a_k h_{n-k} + b_n,$$

where $n \geq k$ and $a_1, a_2, \ldots, a_p$ are constants ($a_k \neq 0$). Such a recurrence is called a *linear recurrence of order k with constant coefficients*. If $b_n = 0$, then we say the recurrence is **homogeneous**. The Fibonacci sequence satisfies a linear homogeneous recurrence relation of degree 2, for example. We'll stick to solving linear homogeneous recurrence relations in this section.

If the initial conditions are disregarded, a recurrence has many solutions. Some of these solutions will be sequences of the form

$$\alpha^0, \alpha^1, \alpha^2, \ldots$$

where α is a number. If we substitute x^k in for a_k in our recurrence above, we get:

Example 1: Consider the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 1$.

Theorem: Suppose that a linear homogeneous recurrence with constant coefficients has **distinct** characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_p$, then if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are constants, every expression of the form

$$a_n = \lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n + \cdots + \lambda_p \alpha_p^n$$

is a solution to the recurrence. This is called the *general solution* of the recurrence.

Example 2: Find the general solution for the Fibonacci recurrence, i.e. find a formula for f_k for $k \geq 0$.

Example 3 A case with multiple roots: Solve the recurrence

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3},$$

where $a_0 = 1, a_1 = 2, a_2 = 0$.

§7.5 **Nonhomogeneous Recurrence Relations**

Recurrence relations are not all homogeneous. For example, consider the following example:

Tower of Hanoi: The Tower of Hanoi consists of three rods and n disks of different sizes which can slide onto any rod. The puzzle starts with the disks neatly stacked in order of size on one rod, the smallest at the top. The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:

- Only one disk may be moved at a time.
- No disk may be placed on top of a smaller disk.

How many moves does it take to move all n disks?

Example 2 (Recurrences and Generating Functions): Use generating functions to solve the recurrence $a_{k+1} = 3a_k + 2, a_1 = 1$, i.e. find a closed form for a_k.

Example 3: Solve $h_n = 3h_{n-1} + 3^n (n \geq 1)$ with $h_0 = 2$.

§8.1 **Catalan Numbers**

The **Catalan sequence** is the sequence $C_0, C_1, \ldots,$ where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} \forall n \in \mathbb{N}$.

Theorem 8.1.1 Let A_n be the number of sequences $a_1, a_2, \ldots, a_{2n}$ of $2n$ terms that can be formed using exactly n $^{+}1$'s and exactly n $^{-}1$'s whose partial sums are always *nonnegative*, i.e. $a_1 + a_2 + \cdots + a_k > 0$ for $1 \leq k \leq 2n$. Then $A_n = C_n$, the n^{th} Catalan number.

Example using Theorem 8.1.1: Suppose there are $2n$ people in a line to get into a theater. Admission is 50¢. Of the $2n$ people, n have a 50¢-piece and n have a 1\$ dollar bill. The box office began with an empty cash register. In how many ways can the people line up so that whenever a person with a 1\$ dollar bill buys a ticket, the box office has a 50¢-piece in order to make change?

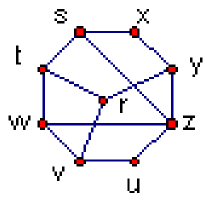
Let $C_n* = (n-1)!\binom{2n-2}{n-1}$.
This sequence satisfies the recurrence $C_n* = (4n-6)C_{n-1}*, n \geq 2$ with $C_1 = 1$ and is called the sequence of **pseudo-Catalan** numbers.

Example (§8.1 pg. 270): Let $a_1, a_2, \ldots, a_n$ be n numbers. A multiplication scheme requires a total of $n-1$ multiplications between two numbers, each of which is either one of $a_1, a_2, \ldots, a_n$ or a partial product of them. Let h_n denote the number of multiplication schemes for n numbers: $a_1 \times a_2 \times \ldots a_n$. Find a formula for h_n.

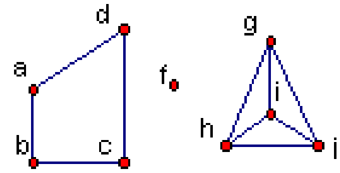
Triangulation of an $(n+1)$-gon:

Chapter 11: Graph Theory

§11.1 **Basic Properties**



G=(V,E)



H=(V,E)

We'll start with a series of definitions.

- A graph G consists of a (usually) finite set V of **vertices** (think points) and a set E of **edges** that connect two (not necessarily distinct) vertices in V.. We write $G = (V, E)$ to denote the graph with vertex set V and edge set E.

  Note that for our purposes (at least initially), edges have no direction associated with them.

- The number of vertices in a graph $G = (V, E)$ is called the **order** of the graph. Note the order of $G = (V, E)$ is $|V|$.

- For a graph $G = (V, E)$, we say two vertices $u, v \in V$ are **adjacent** if $\{u, v\} \in E$. We say a vertex u is **incident** to the edge e if $u \in e$.

- The **degree** of a vertex v is the number of edges incident to it and is denoted $\deg(v)$.

- A **loop** connects a vertex to itself.

- A graph has **multiedges** if between some pair of vertices u and v, there is more than one edge connecting u to v.

- A **simple** graph has <u>no multiedges and no loops</u>. (We will almost exclusively deal with simple graphs.)

- A **multigraph** is a graph which *may have* multiedges or loops (or both). (A simple graph is technically a multigraph.)

- A sequence of (not necessarily distinct) vertices $v_0, v_1, v_2, \ldots, v_m$ is a walk of length m if $\{v_{i-1}, v_i\}$ is an edge for $1 \leq i \leq m$. If $v_0 = v_m$, then we say the walk is **closed**. Otherwise the walk is **open**.

- If a walk has distinct edges, it is called a **trail**. If a walk has distinct vertices (except possibly $v_0 = v_m$), then it is called a **path**.

- A closed path of length $m \geq 3$ is called a **cycle** and is denoted C_m.

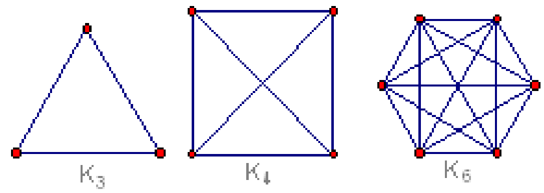
- A graph G is **connected** provided each pair of vertices is connected by a walk. Otherwise, G is **disconnected**.

- Suppose $G = (V, E)$ is a graph. Then $H = (V', E')$ is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$. We write $H \subseteq G$.

- An **induced subgraph** $G[S]$ of a graph $G = (V, E)$ is a graph on the set of vertices $S \subseteq V$ such that if two vertices in S are adjacent in G, then they are also adjacent in $G[S]$.

- If the vertex sets of a graph G and a subgraph H are equal, then we say H is a **spanning subgraph** of G.

- A **complete** graph on n vertices, denoted K_n, is a graph where each pair of vertices is connected by exactly one edge.



Theorem: The number of edges in a complete graph K_n is

Proof(s):

The **degree sequence** of a graph G of order n is $(d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

In-class work:

1. Draw a few graphs (simple graphs or multigraphs). Please do not draw only complete graphs.
2. Given a graph $G = (V, E)$, determine a relationship between the number of edges in the graph $|E|$ and the sum of the elements in the degree sequence $(d_1 + d_2 + \cdots + d_n)$.

Theorem (Relating the Number of Edges to the Sum of the Degrees in a graph G):

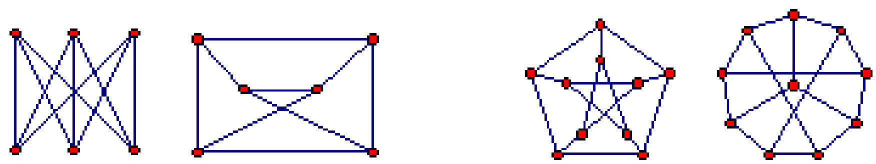
Proof:

Corollary:

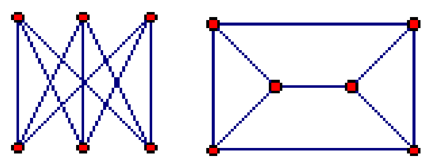
Isomorphic Graphs

Two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are called **isomorphic** if they are "exactly the same, but different." (We write $G \cong H$.) This just means that if you had a drawing program and drew G, you could relabel the vertices (to match H's vertex set) and drag the vertices around to end up at H.

Examples: Each pair of graphs below are isomorphic.



Examples: The graphs below are NOT isomorphic.



Properties of Isomorphic Graphs: Suppose $G = (V, E)$ and $H = (V', E')$ are isomorphic. Here are some obvious properties of G and H:

- G and H have the same order and number of edges (i.e. $|V| = |V'|$ and $|E| = |E'|$).
- G and H have the same degree sequence.
- G is connected if and only if H is connected.
- G has a cycle of length k if and only if H has a cycle of length k.

Exercise: Determine the number of non-isomorphic graphs there are of order 3. Do the same for graphs of order 4.

§11.2 **Eulerian Trails**

Closed Trail Lemma:

Proof:

Euler Circuit Theorem:

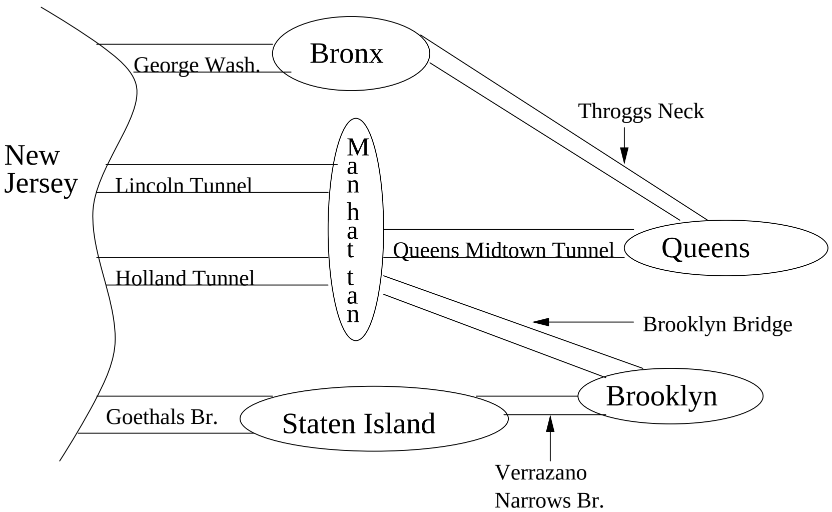
Proof

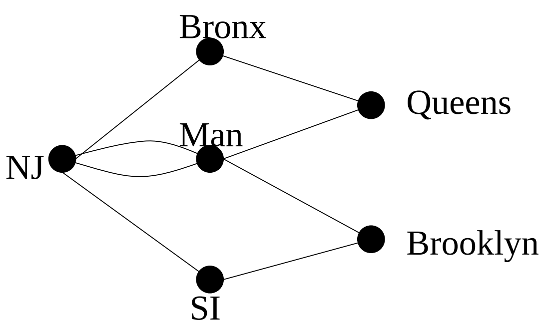
Corollary: Euler Trails:

Proof

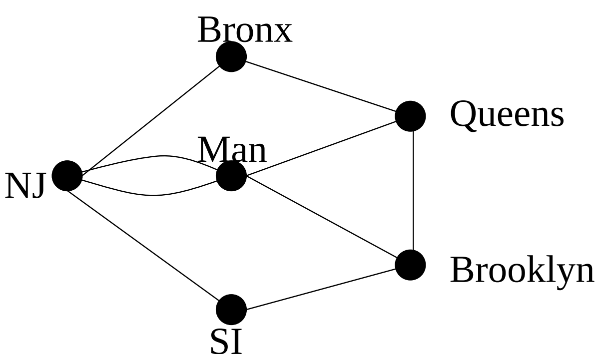
Pizza Delivery Problem

Below is a map of New York City and its surrounding area. Suppose you are a Domino's delivery person in New Jersey. Your boss tells you to deliver a pizza to a toll collector for **each** of the bridges in the map. In order to save time, you want to do this so that <u>you cross each bridge exactly once</u> and return back to New Jersey. Can you do this? If so, show the route you would take. The graph below models the map given.





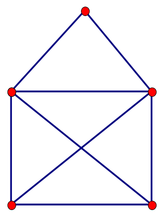
What happens if we add the I-495 (from Brooklyn to Queens) as shown below? Can you start in New Jersey and cross each bridge exactly once? Can you start anywhere else and cross each bridge exactly once?



When you start from a vertex in a graph, walk along every edge exactly once, and return to the starting vertex, we say this graph has an **Euler Circuit**. If you walk along every edge, but do not return to the starting vertex, we say the graph has an **Euler Trail**.

For the following graphs, determine if they have (a) an Euler circuit, (b) an Euler trail (but not an Euler circuit) or (c) neither.

Goal: By looking at a graph, one can tell if it has an Euler circuit/trail or not. What makes a graph have an Euler circuit/trail? **Hint**: Look at the degrees of the vertices.



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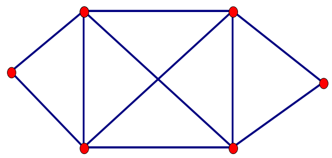

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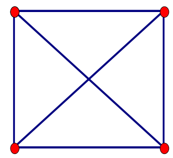

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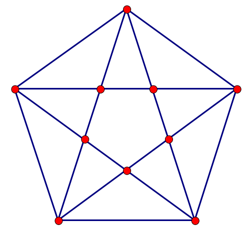

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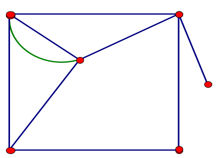

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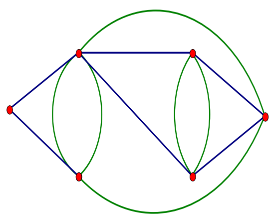

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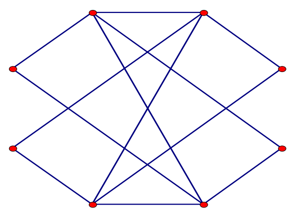

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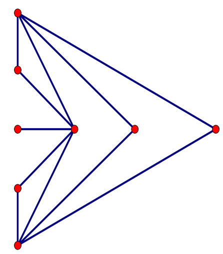

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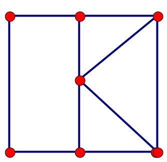

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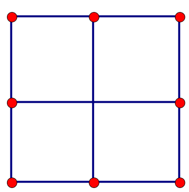

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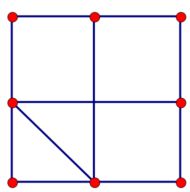

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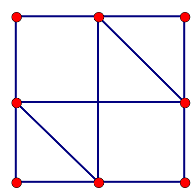

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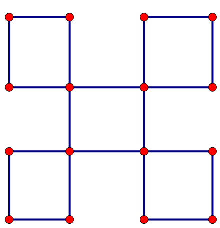

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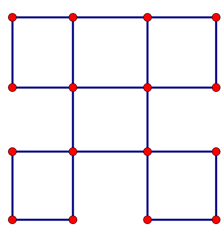

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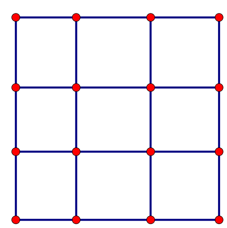

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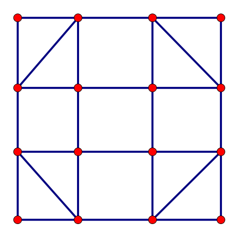


XVI.____

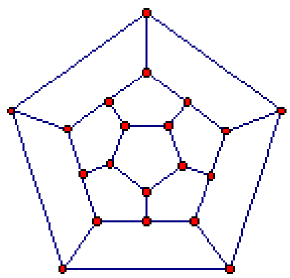
A graph has an Euler circuit when ______________________________________.

A graph has an Euler trail when ____.

§11.3 **Hamilton Path and Cycles**

We say a graph has a **Hamilton cycle** if there is a cycle, which goes through every vertex (not necessarily every edge). If there exists a path that goes through every vertex, then we say the graph has a **Hamilton path**.

Hamilton cycles and paths are named for Sir William Hamilton who marketed a game played on a planar version of the dodecahedron. There is conflicting information about how the game was played. Your book says the object of the game was to start at some corner and return there after visiting every other corner exactly once. *Introduction to Graph Theory* by Douglas West says one player would specify a 5-vertex path, and the other player would have to extend it to a spanning cycle. In either case, the game was not a commercial success because there were too many solutions.

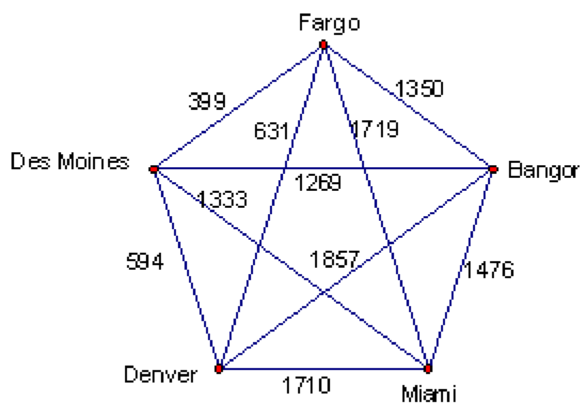


Exercise: How many Hamilton cycles (if any) does K_n have?

Traveling Salesman Problem: The most famous problem involving Hamilton cycles is the Traveling Salesman Problem. Suppose you are a sales*person* and need to visit a bunch of cities. What is the cheapest route you can find that hits all the cities you need to go to and returns you home. Consider the example below:

Example: Suppose you want to go to visit four of your relatives this summer. You are currently living in Fargo, ND. Unfortunately, you cannot afford to fly everywhere, but your car is in good shape. Therefore you will drive to see each of your relatives. Below is a table and graph listing the distances (in miles) between pairs of cities. Try to find the route that requires you to put the least amount of mileage on your car. You must start in Fargo and end up in Fargo.

	Bangor	Denver	Des Moines	Fargo	Miami
Bangor		1857	1269	1350	1476
Denver	1857		594	631	1710
Des Moines	1269	594		399	1333
Fargo	1350	631	399		1719
Miami	1476	1710	1333	1719	

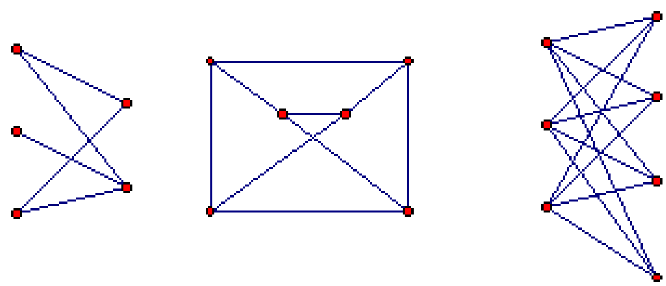


§11.4 **Bipartite Graphs**

A **bipartite** graph is a graph $G = (V, E) = (X \cup Y, E)$ in which the vertex set V can be partitioned into two sets X and Y such that each edge has one vertex in X and one vertex in Y. (i.e. there are no edges between two vertices in X and no edges between two vertices in Y).

A bipartite graph $G = (X \cup Y, E)$ is called **complete** if each vertex in X is adjacent to each vertex in Y (and vice versa). We write $K_{m,n}$ to denote the bipartite graph $G = (X \cup Y, E)$ when $|X| = m$ and $|Y| = n$.

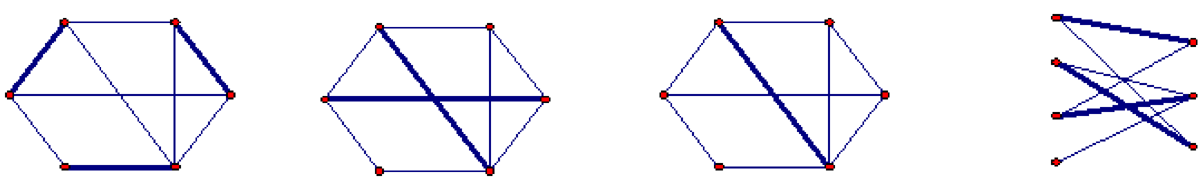
Examples:



Matchings and Bipartite Graphs

Note: this is talked about as **System of Distinct Representatives** (SDR's)'s in your book in Chapter 9.

A **matching** in a graph is a set of pairwise disjoint edges (meaning the edges do not intersect each other). The vertices belonging to the edges of a matching are **saturated** by the matching. The others are **unsaturated**. If a matching saturates every vertex of G, then it is a **perfect matching**.

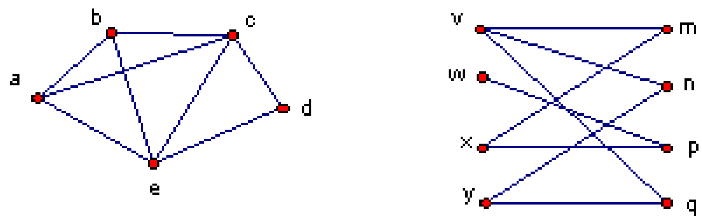


A **maximal matching** M is a matching for $G = (V, E)$ such that $M \cup e$ is no longer a matching for any edge $e \in E \backslash M$. A matching for $G = (V, E)$ is **maximum** if it is largest possible matching.

Examples: See above

Exercise: For what values of n does K_n have a perfect matching?

Some definitions:
Suppose you have a graph $G = (V, E)$ and $A \subseteq V$.
The **neighborhood** of A is $N(A) = \{v \in V : \{a, v\} \in E\}$, i.e., the set of vertices adjacent to at least one vertex in A.



A bipartite graph $G = (X \cup Y, E)$ has an X**-perfect matching** if there exists a matching M that saturates X.

Hall's Marriage Theorem (1935): If $G = (X \cup Y, E)$ is a bipartite graph, then G has an X-perfect matching if $|N(A)| \geq |A|$ for all $A \subseteq X$.

Why does this make sense?

Proof:

A graph $G = (V, E)$ is k**-regular** if $\deg(v) = k$ for all $v \in V$.

Exercise: Suppose $G = (X \cup Y, E)$ is a k-regular bipartite graph. Prove $|X| = |Y|$.

Corollary to Hall's Theorem: A k-regular bipartite graph $G = (X \cup Y, E)$ has an X-perfect matching, and therefore a perfect matching.
Proof:

Stable Marriage Theorem

Suppose we have a collection of n men and n women and want to establish a collection of "stable" marriages. A collection of marriages is **stable** if and only if there is no man x and no woman a such that x prefer a to his spouse and a prefers x to her spouse. Otherwise, the marriage is unstable because a and x would leave their spouses to be with each other.

Example: Given men x, y, z, w and women a, b, c, d and preferences listed below, the matching $\{xa, yb, zd, wc\}$ is a stable matching.

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

Exercise: Find another stable matching in the example above.

Gale and Shapley (1962) proved that a stable matching always exists in a paper entitled "College admissions and the stability of marriage."

Gale-Shapley Proposal Algorithm to Produce Stable Marriages:

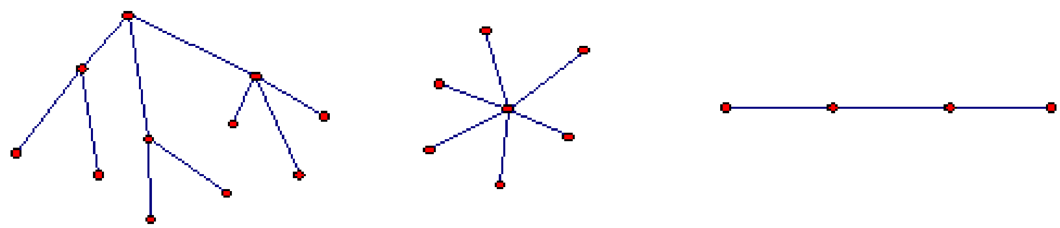
- Each man proposes to the highest woman on his list who has not previously rejected him.
- If each woman receives exactly one proposal during a round of proposals, then each proposal is accepted and everyone lives happily ever after.
- Otherwise, at least one woman received more than one proposal (and some woman receives no proposals....so sad...so sad).
- Every woman receiving more than one proposal rejects all but the highest on her list.
- Every woman receiving a proposal says "maybe" to the most attractive proposal received. (This means a man may propose to the same women repeatedly until she rejects him or is "stuck" with him.)

Proof that the Gale-Shapley Algorithm produces a set of stable marriages:

§11.5 **Trees**

The **components** of a graph are the maximal connected subgraphs. Therefore, two components are never connected.

A **tree** is a connected graph such that the graph becomes disconnected with removal of any edge.



Theorem The following are equivalent for a connected graph $G = (V, E)$ on n vertices:

(a) G is a tree.

(b) G has $n - 1$ edges.

(c) There exists a unique path between each pair of vertices.

(d) G does not have any cycles.

Proof:

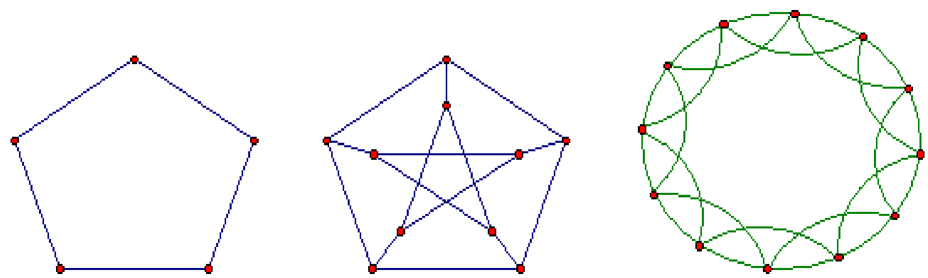
Theorem Every tree has at least two leaves (i.e., two vertices of degree 1).
Proof:

§12.1 **Chromatic Number**

A (proper) **vertex coloring** of a graph $G = (V, E)$ is an assignment of a color to each $v \in V$ such that if $\{u, v\} \in E$, then u and v are assigned different colors.

If the number of colors used in a proper coloring is k, then we say it is a k-**coloring** and the graph is k-colorable.

The smallest k such that a graph G is k-colorable is called the **chromatic number** of G and is denoted $\chi(G)$.



Exercise: Find the chromatic number, i.e., $\chi(G)$, for K_n and C_n.

A few easy theorems (We will only prove the last one):

1. If $H \subseteq G$, then $\chi(H) \leq \chi(G)$.
2. For any graph G of order n, we have $1 \leq \chi(G) \leq n$.
3. A graph G with at least one edge is bipartite if and only if $\chi(G) = 2$.

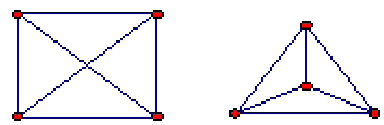
Greedy Coloring Theorem: Let G be a graph graph with maximum degree $\Delta(G)$. Then $\chi(G) \leq \Delta + 1$.
Proof:

Brooks Theorem: Suppose G is a connected graph with maximum degree $\Delta(G)$. Then $\chi(G) \leq n$, as long as G is not an odd cycle and G is not K_n.

§12.2 **Plane and Planar Graphs**

A **planar graph** is a graph that *can* be drawn in the plane with no edges crossing. When is a planar graph G is drawn with no edges crossing, we call that a **plane graph** or a **planar representation** of G.

Example: Below is a non-planar and a planar representation of K_4, (a planar graph).

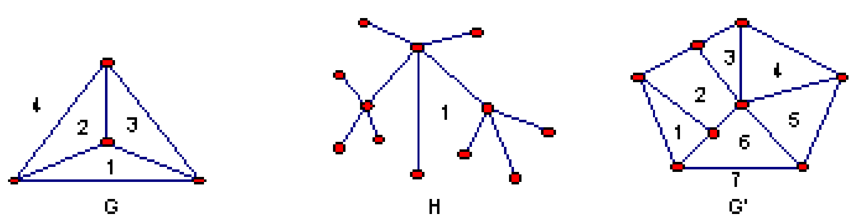


Theorem: Every planar graph has a planar representation where all the edges are straight.

Examples: Draw a few plane graphs below.

A **face** of a plane graph is either (i) a cycle with no internal vertices or (ii) the outside face which you can think of a fence around the entire graph.

Examples: Each of the graphs below has the faces of the graph labeled.



Euler's Formula (for planar graphs): If G is a planar graph with e edges, n vertices and f faces, then:

Two Similar Theorems: A complete graph K_n is planar if and only if $n \geq 4$. A complete bipartite graph $K_{m,n}$ is planar if and only if $m \leq 2$ or $n \leq 2$.

Proof:

§12.3 **A Five-Color Theorem**

A Famous Solved Problem with an AWFUL, UGLY, TOO LONG Proof, but a Great Answer: Suppose you are given a blank map (which of course is planar). How many colors would you need to color the regions of the map if two regions sharing a border must use different colors?

The Five-Color Theorem: The chromatic number of a planar graph is at most 5.