Final Exam from 2007 Solutions

1. Suppose that we have 10¢, 18¢, and 28¢ stamps, each in unlimited supply. Let $s(n)$ be the number of ways of obtaining n cents of postage if the order in which we put on stamps counts. For example, $s(10) = 1$ and $s(20) = 1$ (two 10¢ stamps), while $s(28) = 3$ (one 28¢ stamp, or a 10¢ stamp followed by an 18¢ stamp, or an 18¢ stamp followed by a 10¢ stamp).

   (a) If $n > 29$, derive a recurrence for $s(n)$. (Do not solve the recurrence.)
   $s(n) = s(n-28) + s(n-18) + s(10)$. This is because you can either begin with a 10¢, a 18¢, or a 28¢ stamp.

   (b) Use the recurrence of part (a) to find the number of ways to find the number of ways of obtaining 66¢ of postage.
   $s(66) = s(38) + s(48) + s(56)$

   $s(38) = s(10) + s(20) + s(28) = 1 + 1 + 3 = 5$

   $s(48) = s(20) + s(30) + s(38) = 1 + (s(2) + s(12) + s(20)) + 5 = 1 + (0 + 0 + 1) + 5 = 7$

   $s(56) = s(28) + s(38) + s(46) = 3 + 5 + (s(18) + s(28) + s(36)) = 8 + (1 + 3 + (s(8) + s(18) + s(26))) = 12 + 0 + 1 + 0 = 13$

   Therefore $s(66) = 5 + 7 + 13 = 25$.

2. We print one 5-digit number on a slip of paper. We include numbers beginning with 0's; for example, 00158. Since the digits 0,1, and 8 look the same upside down, and since 6 and 9 are interchanged when a slip of paper is turned upside down, 5-digit numbers such as 61891 and 16819 can **share the same** slip of paper. If we want to include all possible 5-digit numbers but allow this kind of sharing, how many different slips of paper do we need?

   Note that the only way a number is printed on the same piece of paper as another number is if (1) the digits of the number come from $0, 1, 6, 8$ or 9 only AND (2) your number is not the same as itself upside-down.

   The numbers satisfying (2) are of the form $abcba$, where $a, b \in \{0, 1, 6, 8, 9\}$ and $c \in \{0, 1, 8\}$. Note we are not being completely truthful here because although 0,1, or 8 is the same as itself upside-down, we have 6 is the same as 9 upside down. In either case, we could say the number is really of the form $abcb'a'$ where a and a' are the same upside-down, as are b and b'. Such a number looks like 01010 or 01810 or 69169 or 61819.

   To solve this problem, we'll take all possible 5-digit numbers and get rid 1/2 of the numbers that appear on the same piece of paper as another number. The number of all possible 5-digit number is 10^5.

   The number of possibilities for upside down numbers is 5^5 because you need any of the numbers in the set $\{0, 1, 6, 8, 9\}$ as the 1st, 2nd, 3rd, 4th, and 5th digits. Then you have to get rid of the ones satisfying (2) above. The number of self-upside-down 5-digit numbers is $5^2 \cdot 3$ because the 1st two digits determine the last two digits. Again any of the numbers in the set $\{0, 1, 6, 8, 9\}$ are your 1st and 2nd digits and any member of the set $\{0, 1, 8\}$ as your 3rd digit.

   Therefore your total number of slips of paper is $10^5 - \frac{1}{2}(5^5 - 5^2 \cdot 3) = 100000 - 1525 = 98475$.

3. A maximal matching in a graph is a matching such that no other edges can be added. (Note that maximal does not necessarily equal maximum.) Determine (with proof) the minimum size of a maximal matching in the cycle C_n. (This means find a formula for this minimum size depending on n.)

   Hint: It has to do with what n is (mod 3), i.e., it depends on the remainder when n is divided by 3.

4. A Hamiltonian cycle in a graph $G = (V, E)$ where $|V| = n$ is a cycle of length n, i.e. a cycle that goes through all of its vertices.

We played around with Euler Circuits and Euler Trails in class. It ends up that if $G = (V, E)$ is a connected graph, then:

G has an Euler Circuit if and only if every vertex of G is even.

G has an Euler Trail if and only if exactly two vertices of G are of odd degree.

(a) How many Hamilton cycles are there in $K_{n,n}$?

$\dfrac{n!(n-1)!}{2}$.

(b) For what values of n does K_n have an Euler Circuit?

n odd because you need each degree to be even and the degree of each vertex is $n - 1$

(c) For what values of n does $K_{n,n}$ have an Euler Circuit?

n even because you need each degree to be even and the degree of each vertex is n.

(d) For what values of n does $K_{n,n}$ have an Euler Trail?

You need two odd degree vertices and the degree of each vertex is n. This means we either have $2n$ odd or $2n$ even degree vertices. Thus if we need two odds, we need $n = 1$.

5. In a league with two divisions 13 teams each, determine whether it is possible to a schedule a season with each team playing nine games against teams within its division and four games against teams in the other division. If so, give a way to do it. If not, explain why not.

Recall the theorem that there has to be an even number of vertices of odd degree in any graph. Set up a graph for this problem.

6. (a) Use **combinatorial reasoning** to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

This was in your homework in a previous assignment.

(b) Use **combinatorial reasoning** to derive the identity

$$n! = \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \cdots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0$$

Here D_n is the number of derangements of an n-set, and D_0 is defined to be 1.

The LHS is the number of permutations of an n-set. Note each permutation has k fixed items and $n - k$ deranged items for some $0 \leq k \leq n$. The number of ways to choose the k fixed items is $\binom{n}{k}$. The number of ways to derange the other $n - k$ items is D_{n-k}. Therefore

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} = \text{ RHS.}$$

7. (a) A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

Proof by the Pigeonhole Principle. For each set A, let $A' = \{1, 2, \ldots, n\}\ A$ be its complement. Form the set of 2^{n-1} pairs (A, A'). The reason why we have 2^{n-1} such pairs is that each set has a unique complement and there are 2^n subsets of our set $\{1, 2, \ldots, n\}$, we take $\frac{2^n}{2} = 2^{n-1}$.
Suppose there were more than 2^{n-1} subsets in the collection of subsets of $\{1, 2, \ldots, n\}$ with the property that each pair of subsets has at least one element in common. That would mean, by the pigeon hole principle (the subsets are the pigeons and the pairs of subsets are the pigeonholes), that a subset and its complement have been chosen. This contradicts that each pair of subsets has at least one element in common.

(b) Prove that

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$$

$$1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\frac{n!}{k!(n-k)!} = \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!(n-k)!} = n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$$

Using $j = k - 1$, we get

$$n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} = n\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} = n2^{n-1}$$

by the Binomial Theorem.

(c) How different ways are there to distribute 8 (distinguishable) gifts to 5 (distinguishable) children,

i. if it is not necessary that each kid receive at least one gift?

5^8.

ii. if each child must receive at least one gift? (Hint: Use the Principle of Inclusion-Exclusion, where you use part (a) and deal with the cases where one kid does not receive a gift, two kids don't receive gifts....)

If one kid does not receive a gift, then we choose which kid it is $\binom{5}{1}$ and distribute the gifts to the other 4 kids 4^8: $\binom{5}{1} \cdot 4^8$.

If two kids do not receive a gift, then we choose which kids $\binom{5}{2}$ and distribute the gifts to the other 3 kids 3^8: $\binom{5}{2} \cdot 3^8$.

If three kids do not receive a gift, then we choose which kids $\binom{5}{3}$ and distribute the gifts to the other 2 kids 2^8: $\binom{5}{3} \cdot 2^8$.

If four kids do not receive a gift, then we choose which kids $\binom{5}{4}$ and distribute the gifts to the other 1 kid: $\binom{5}{4}$.
The total number of ways so that each kid receives a gift, by the Pigeonhole Principle, is

$$5^8 - \binom{5}{1} \cdot 4^8 + \binom{5}{2} \cdot 3^8 - \binom{5}{3} \cdot 2^8 + \binom{5}{4} = 127280$$