

# Unpacking Division to Build Teachers' Mathematical Knowledge

"First I broke apart 169 into 140 and 29 and then I divided 140 by 14," explained Andreau. Kendra commented that she also used 140, but began by multiplying  $10 \times 14$ . These teachers are describing the approaches they used to solve  $169 \div 14$ . They were challenged to generate two strategies for solving this multidigit division problem without using the conventional division algorithm. While examining their work, we were struck by their varied approaches to the task and by the operations that their solution strategies encompassed. We were intrigued when they became stuck in their thinking and when they gained insights into the current boundaries of their division knowledge.

Based on our experience working with both prospective and practicing teachers, we have found a need to support teachers in developing a much deeper understanding of number and operations, particularly for division. Our knowledge as adults is often compressed (Ball and Bass 2000). In fact, we have worked for years to compact and streamline our mathematical knowledge. This compression of knowledge is central to the discipline of mathematics; however, "knowing mathematics sufficiently for teaching requires being able to unpack ideas and make them accessible as they are first encountered by the learner, not only in their finished form" (Ball 2003, p. 4). Prospective teachers enter our courses with compressed knowledge of division. They are able to demonstrate procedural or algorithmic skill with division of multidigit

numbers, but they are limited in their foundational, conceptual understanding, which was long ago compacted or perhaps never even known.

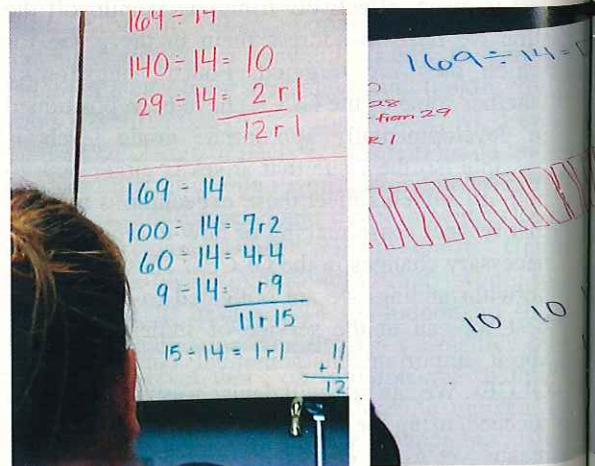
The intent of this article is to examine the unpacking of the mathematical knowledge necessary for teaching division. What does one need to know and understand in order to teach division well? What makes up the package (Ma 1999) or bundle (Ball and Bass 2000) of knowledge—the key ideas, understandings, connections, and sensibilities—that teachers must develop so that it is available for teaching? How can we, as teacher educators, reveal the complexities of division in ways that support teacher learning? We first present a core task for surfacing and unpacking one's division knowledge and then discuss the understandings that might comprise a package of teacher knowledge for division.

## Core Task

The purpose of a core task is to reveal teachers' current knowledge and understandings and to provide a context for grounding discussions over several class sessions. Asking teachers to generate strategies for solving multidigit division problems has been effective for us in probing what teachers know about division. We gave them the following task:

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The purpose of this task is to begin to think more deeply about division, so put your thoughts, attempts, and missteps all on paper. Solve the following division problems using two strategies other than the conventional division algorithm. Then solve the problems using the algorithm. Explain and represent your thinking using symbols, words, and diagrams, as appropriate.

$$169 \div 14$$

$$3480 \div 36$$

As the Conference Board of the Mathematical Sciences (2001) recommends in its report *The Mathematical Education of Teachers*, the key to turning prospective teachers “into mathematical thinkers is to work from what they do know—the mathematical ideas they hold, the skills they possess, and the contexts in which these are understood—so they can move from where they are to where they need to go” (p. 17). We have found reflection on and discussion of one’s own mathematical work to be a valuable teaching tool in building one’s mathematical knowledge for teaching.

We purposely selected the division problems in this core task to establish a basis on which to connect and deepen one’s knowledge. We chose  $169 \div 14$  because of its potential to prompt the use of multiples of 10 and because it can be solved readily using direct modeling or repeated subtraction. We chose  $3480 \div 36$  to prompt more complex strategies because the numbers do not lend themselves well to using a multiple of 10. We also chose problems with remainders to see their impact on solution strategies.

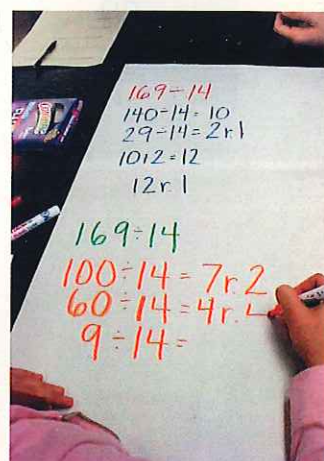
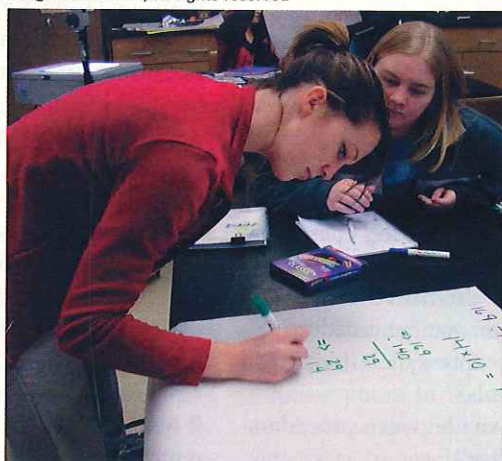
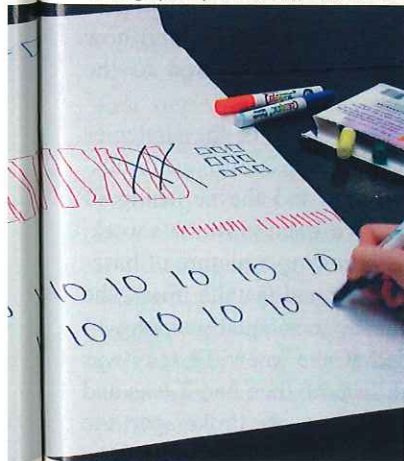
We hoped to accomplish several objectives by posing this task. First, we wanted to gain insight into teachers’ grasp of number sense and operation sense. In particular, we wanted to see if they could demonstrate relationships among the operations. Second, we knew that they could carry out the conventional division algorithm, but we wondered how deeply they understood it and whether they realized its complexity. We planned to use their solution strategies to tie together and connect pieces of mathematical knowledge while generating an appreciation of the complexity of the algorithm. Finally, we wanted the teachers to increase their own flexibility in solving multidigit division problems by developing the ability to use a variety of computational strategies.

## Reviewing and Selecting Work Samples

Our general practice is to assign the task as homework two class sessions prior to discussing it. This gives us time to carefully study and reflect on the teachers’ work. We review the papers once, looking for certain themes. Questions we consider during the first review include the following:

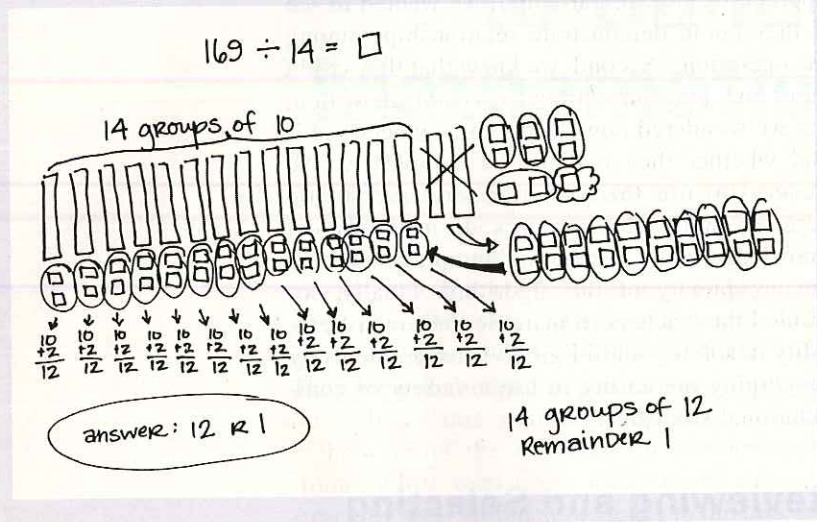
- How successful was the class in devising alternative strategies that demonstrate teachers’ understanding of division?
- Is there a “preferred” alternative strategy demonstrated by the class as a whole?
- What collective mathematical struggles or insights surfaced?
- Were there surprises or curiosities?

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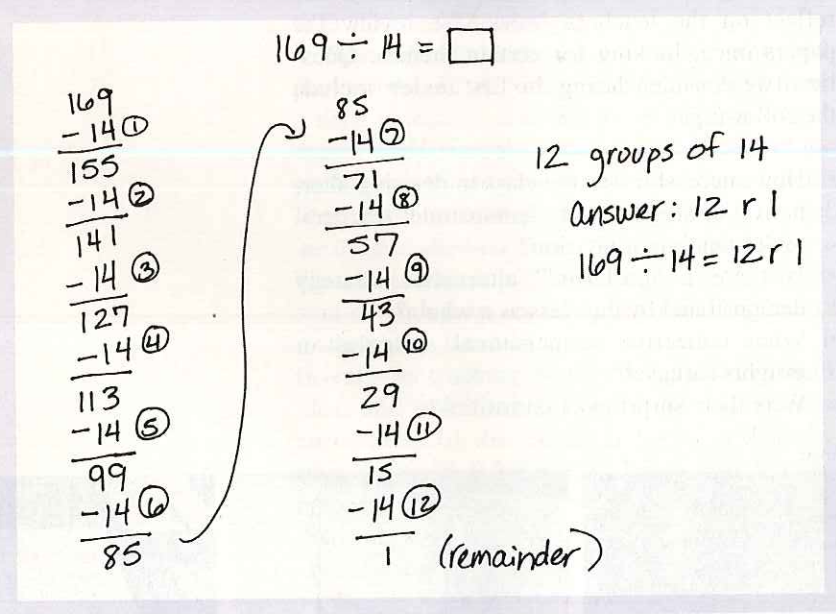
**Figure 1**

Dealing out 169 into 14 groups



**Figure 2**

Using subtraction as a related operation to solve 169 ÷ 14



A second look through the papers encourages a more careful review with the following questions:

- How fluent are teachers' alternative strategies?
- How accurate is the mathematical notation?
- Are the teachers able to express their thinking in writing as well as symbols?
- Do solid connections exist between procedural and conceptual knowledge?

- Did specific individuals demonstrate mathematical insights or misconceptions in their work?
- What "windows" are offered into an individual's thinking or struggles?

Finally, we select specific work samples for class discussion and examination. In the past, we have written the strategies on the chalkboard or made overhead transparencies. Our preference now is to put the selected strategies on chart paper. This allows us the opportunity to discuss each strategy on its own and to place strategies side by side for comparison. We leave it up to the teachers to decide if they would like their name attached to their strategy or if they wish to remain anonymous.

## Unpacking Division by Examining Strategies

Our goals for discussion include a careful examination of the embedded mathematics in the various strategies and the use of mathematical notation. We also want to give the teachers opportunities to practice providing explanations, following the logic of one another's strategies, and using the strategies on new problems. For many prospective teachers, the task is an eye-opening experience as they work with numbers in ways they had never before considered. Their reactions range from "I had never thought division problems could be solved in so many different ways" to "I couldn't get any ways to work successfully."

To bridge this range of knowledge, we provide a structured process for examining the selected strategies. First, we display a selected work sample. Then, in pairs, teachers take turns explaining what they see occurring in the algorithm and paraphrasing the other's explanation. This creates the foundation for an in-depth, whole-class discussion of the embedded mathematics and how the alternative strategies are connected to the original problem.

We now briefly present some of the strategies for 169 ÷ 14 to provide a glimpse into our discussions of the work samples and the surfacing of mathematical ideas. **Figure 1** shows Maria's work. She represented 169 by drawing a picture of base-10 blocks showing 16 tens and 9 units. First, she distributed or dealt out the tens equally among 14 groups. Maria stated that she knew 14 tens was equivalent to 140 and that she then had 2 tens and 9 ones left to distribute. Next, she broke apart the remaining 2 tens into 20 ones and distributed the

ones equally among the 14 groups. The meaning of division in partitive situations emerged from discussion of this work.

We compared Maria's work to the work in **figure 2**, which shows the use of repeated subtraction and an interpretation of division as a measurement situation. This approach surprised many of the teachers because they had never established a relationship between division and subtraction prior to this task. A particular highlight of this discussion was the identification and comparison of the meaning of the numbers 169, 14, 12, and 1 in each situation and why the meanings differed. In this case, we found creating a real-world context for both the partitive and measurement models to be helpful in pushing for deeper understanding. That interpretations could be so different, yet still represented by the same equation, was intriguing.

Kendra's work (see **fig. 3**) illustrates the use of a missing-factor approach. She explained, "I started by thinking about a multiplication problem using 14 that would get me close to 169. I used 10 because it's easy to multiply:  $10 \times 14 = 140$ . That gives me 10 groups of 14." Then she explained how she added on additional groups of 14 until she got as close to 169 as she could without going over. This discussion encouraged a closer look at the connections between multiplication and division and pushed us to examine the belief that "division is the opposite of multiplication." It also served as the basis for highlighting division as the inverse of multiplication as well as for making a connection to ways in which addition could be used as a related operation. Kendra reasoned with a measurement interpretation of division by finding the number of groups of size 14. This led to an interesting comparison to others in the class who used a missing-factor approach but reasoned with a partitive interpretation of division.

As **figure 4** shows, Andreau partitioned the dividend into  $140 + 29$ . Her reasoning behind this first step was to "break apart" the dividend into two numbers, with one being a number easily divisible by the divisor. Then she combined the partial quotients. This work prompted a discussion of decomposition of numbers and properties of the operations. The following question emerged: "If one can partition the dividend and it works, why can't one partition the divisor?" Several teachers had attempted to do just this and discovered that it did not seem to work. They had partitioned 14 into  $10 + 4$ , then divided 169 by 10, then divided 169 by 4. The ensuing discus-

**Figure 3**

Using multiplication as a related operation to solve  $169 \div 14$

$$169 \div 14 = \square$$

$$14 \times 10 = 140$$

$$14 \times 2 = 28$$

$$10 + 2 = 12 \text{ (12 groups of 14)}$$

$$140 + 28 = 168$$

$$169 - 168 = 1 \text{ (remainder)}$$

answer: 12 r 1

$$169 \div 14 = 12 \text{ r } 1$$

sion engaged small groups in development of arguments about whether one could partition the divisor. Although some groups tried to generate a formal proof using properties of the operations, situating the problem in a real-world context proved more convincing. For example, assume you have 169 stickers to share among 14 students, 10 girls and 4 boys. If you decompose the divisor, you will be sharing 169 stickers among 10 girls and then sharing 169 stickers among 4 boys, thus creating a different situation.

## A Package of Division Knowledge for Teachers

In order to identify the essential pieces that compose a package of division knowledge for teachers, we reviewed reports including *The Mathematical Education of Teachers* (CBMS 2001), *Adding It Up* (NRC 2001), and *Principles and Standards for School Mathematics* (NCTM 2000), examined textbooks in the field, and reflected on our experiences with teachers and elementary school students. Although we believe that creating a definitive list of topics is impossible, this gave us a starting point. In identifying topics, we attempted to discern the unique subject-matter knowledge necessary for teachers by examining how the math-

**Figure 4**

Partitioning the dividend into 140 + 29

Handwritten mathematical work showing the partitioning of 169 into 140 and 29, and the subsequent division of each part by 14:

$$\begin{array}{l} 169 \div 14 = \square \\ \swarrow \quad \searrow \\ 140 + 29 \end{array}$$
$$140 \div 14 = 10$$
$$29 \div 14 = 2 \text{ R}1$$
$$10 + 2 = 12$$
$$169 \div 14 = 12 \text{ R}1$$

emathical knowledge would be used in the practice of teaching (Ball and Bass 2000).

The topics in **figure 5** adjacent to the word *division* (shaded in blue) formed our initial list. Teachers must understand division as the inverse of multiplication as well as its relationships to addition and subtraction. Teachers also must be familiar with interpretations, models, representations, and applications of division. Properties of the operations and decomposition of numbers also appear to be important knowledge pieces.

From this, we added the topics around the outside (shaded in yellow) as a result of our interactions with prospective teachers and the knowledge that emerged as important for this particular group, such as giving meaning to the numbers and mathematical notation. As this package emerged, we noted that our initial topics were phrased as declarative statements. This is the knowledge we wanted the prospective teachers to learn. The topics that we added were action statements more reflective of mathematical knowledge necessary in teaching.

Another important understanding for teachers is the connectedness of knowledge. Teachers “need to see the topics they teach as embedded in rich networks of interrelated concepts [and] know where, within those networks, to situate the tasks they set their students and the ideas these tasks elicit” (CBMS 2001, p. 55). The connections among these ideas, concepts, and procedures enable teachers to portray mathematics as a unified body of knowledge rather than as isolated topics (Ma 1999).

Knowing and being sensitive to all these components of the package is the work of becoming a teacher of mathematics. The teacher will then be better able to draw on and make strategic use of these ideas in the practice of teaching (Ball 2003; Ball and Bass 2000). We view this package of division knowledge as a work in progress and plan to continue refining the package through further inquiry and work with teachers.

## Important Knowledge Pieces

Within any package of knowledge, certain pieces hold more significance than others (Ma 1999). The most important understandings for these prospective teachers were the relationships among the operations. As one teacher noted, “I am amazed that multiplication, subtraction, and addition can all help me get to the answer.” Another individual more representative of the group commented, “I think I always understood the connections between all the operations and division, but I have never actually used them to solve division problems before.” Ma (1999) described the relationships among the four basic operations as the “road system” connecting all of elementary mathematics. We were surprised by the initial inclination of some teachers to not use these relationships in their strategies. Eventually, the use of related operations, particularly multiplication, became the strategy of choice among this group of teachers.

Another important understanding was a deeper connection to the meaning of division. Most of the prospective teachers admitted they could do the conventional division algorithm, but they remarked that they “never really understood what was going on or even how to explain it” and that “coming up with other ways to solve division problems helped to see what division really means.”

A related important sensibility was an inclination to visualize actions and situations for division as reflected in the following comment: “I started to look at division like I look at multiplication—seeing things in groups.” This supported the ability to identify the meaning of the numbers and to follow the logic of one another’s strategies.

Finally, the prospective teachers developed a greater appreciation of varied strategies and of the connection between their own mathematical

**Figure 5****Example of a package of division knowledge for teachers**

Use and understand varied strategies and algorithms	Flexible decomposition of numbers	Division as the inverse of multiplication	Relationships among addition, subtraction, multiplication, and division	Connect informal and formal mathematical language and notation
Establish equivalence of representations and strategies	Models for division (set, area, array, linear, branching)	<b>DIVISION</b>	Interpretations of division (measurement, partitive, combinations)	Give meaning to the numbers (different or same units)
Judge accuracy of strategies and potential for generalization	Representations (physical, geometric, symbolic)	Properties of the operations	Applications and real-world situations for division	Understand varied meanings of remainders
Use multiplicative relationships	Visualize actions and situations	Pose situations with varied problem structures	Identify use of place value	Grasp of division by zero as undefined

knowledge and their future role as classroom teachers. As one teacher reflected, "Seeing others' points of view and different ways of thinking will help me in realizing that although I understand something one way, students may not."

## Closing Comments

The mathematical knowledge necessary for teaching differs from that necessary for other occupations. Mathematical knowledge must be usable in the practice of teaching, whether in selecting instructional tasks, representing ideas, orchestrating a class discussion, or evaluating students' verbal or written responses (Ball 2003). This requires unpacking compressed knowledge and bringing it to the surface for examination.

One way to surface this knowledge is to engage teachers in core tasks and then use their own work as sites for discussion. In particular, we have found that asking teachers to generate their own alternative computational strategies is productive in revealing strengths and gaps in their current understandings. Then their packages of mathematical knowledge can be further developed and become more accessible as a mathematical resource for teaching.

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