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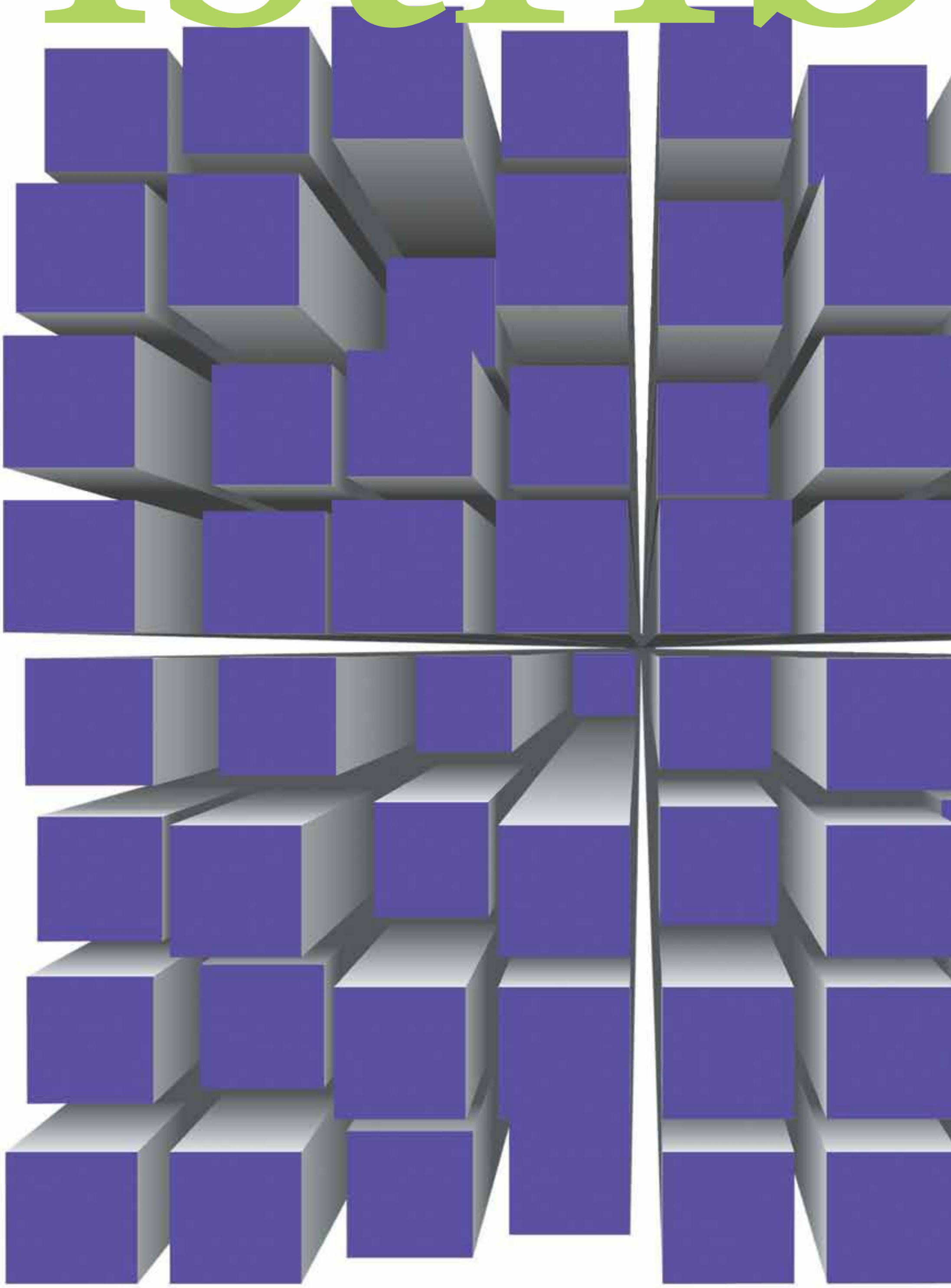

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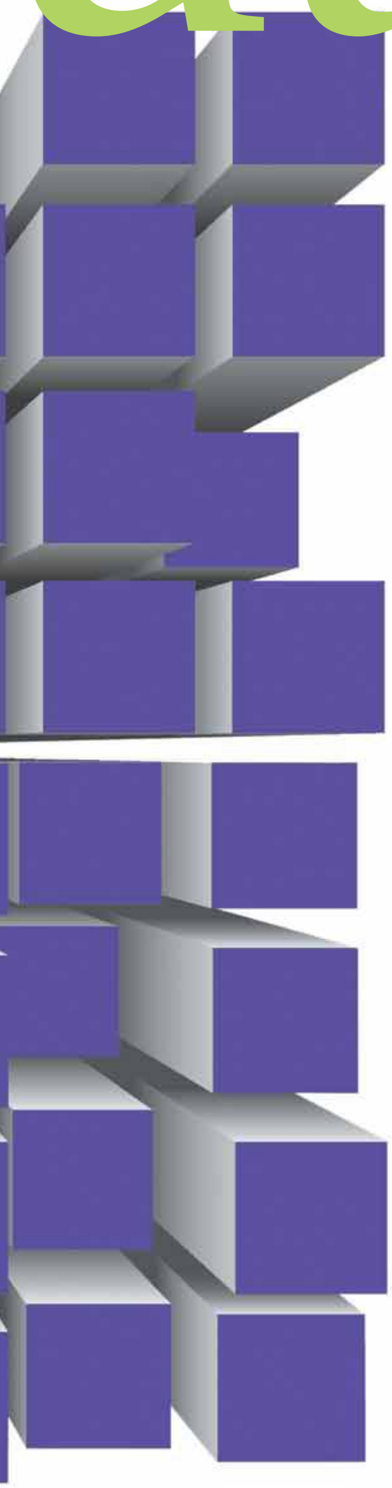

utive Property:

The Core of Multiplication

This activity sequence illustrates the conceptual development of important mathematical ideas, among them the understanding of area and how to deconstruct complicated problems.

By Cathy J. Kinzer and Ted Stanford

One of the four critical areas for third grade in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010), is an understanding of multiplication and division. At least half of all the third-grade Standards in CCSSM involve multiplication in one way or another. Students are expected to memorize facts, use models for multiplication, and use their knowledge of multiplication to explain patterns and solve applied problems. Although there is much to learn, some big mathematical ideas can connect different aspects of multiplication.

The distributive property (DP) is one such big idea. It may be the single most powerful and useful concept related to multiplication. In the past, the distributive property has often been thought of as a property that one studies in high school algebra—something too abstract and advanced for the primary grades. However, the DP is present from the beginning steps of learning multiplication. In its simplest forms, the DP is as easy and intuitive as understanding basic one-digit multiplication facts.

The third-grade Standards reference the distributive property twice, so clearly we are to regard it as an important part of multiplication for beginning learners. Many countries introduce the property, both as a concept and as a computational strategy, in the primary grades (Cai and Knuth 2011; Ding and Li 2010).

We present a sequence of learning activities (available in the **online** appendix to this article) that lead to using the area model of multiplication to understand the DP. The connection between area and multiplication is an important one, both for algebraic thinking and for geometry, as indicated in two of the critical areas for the third grade in CCSSM (2010, p. 21):

1. Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-size groups, arrays, and area models.
2. Students recognize area as an attribute of two-dimensional regions. . . . By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication.

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Students often use addition to begin to make sense of multiplication. They learn that 2 × 3 is the same as 2 threes or 3 + 3. However, relying on the repeated addition concept alone is not enough to fully understand multiplication (Devlin 2011; Yoshida 2009). Consider a student who is trying to remember the answer to 6 × 8. One strategy is to count by eights up to the sixth number—8, 16, 24, 32, 40, 48—which takes time and concentration as well as invites mistakes. A more efficient strategy is to say, "I know that 5 × 8 = 40, so 6 × 8 is just one more eight." This kind of reasoning is the first step in moving beyond repeated addition and using the distributive property to make sense of multiplication.

Students who make friends with the DP early on will find that it is a friend for life. It begins by helping them to understand basic whole-number multiplication. Moving up the grades, it will be an invaluable tool for understanding multiplication of fractions, mixed numbers, negative numbers, and algebraic expressions. We think of the distributive property as the "Common Core Idea of Multiplication"—the property that is present in almost any situation where multiplication is being studied or used.

Defining the property

The following facts are for teacher reference; they are not meant to be part of the series of steps of understanding for children. The distributive property—

- tells you how to add one to a factor in a multiplication problem. For example, if you know that 5 × 4 = 20, then you can figure out 6 × 4 by adding just one more 4 (Yoshida 2009).
- shows you how to break down a complicated multiplication problem into simpler problems. Just as the number 7 can be decomposed into 5 and 2, the problem 7 × 8 can be decomposed into 2 × 8 and 5 × 8.
- is the foundation of the rules for multiplying by zero and by negative numbers (7.NS.2.a).
- gives the answer to why 1/2 × 8 is the same as 8 ÷ 2. We know that 1/2 + 1/2 = 1, so then (1/2 + 1/2) × 8 = 8. But by the DP, (1/2 + 1/2) × 8 = (1/2 × 8) + (1/2 × 8). So, whatever 1/2 × 8 is, we know that two of them together make 8. So, we have to divide 8 by 2, and the answer is 4. We can use the same argument to understand multiplication by other fractions.
- is an informal mental computation strategy for many adults. For example, in calculating a 15 percent tip on a $24 restaurant tab, one could figure that 10 percent is $2.40, and so 5 percent is $1.20. Add the two to get a tip of $3.60. Written as a mathematical equation, it is just the DP: (0.15 × 24) = (0.1 × 24) + (0.05 × 24).
- has a meaningful interpretation in the context of multiplication problems used

in many formulas in science and other subjects. For example, consider the equation distance = rate × time. If you travel at 60 mph for three hours, and then you travel at 60 mph for two more hours, then you have traveled at 60 mph for five hours. We can write this in the form of the DP: $(60 \times 3) + (60 \times 2) = (60 \times 5)$.

- may be written as $a \times (b + c) = (a \times b) + (a \times c)$. This equation is powerful because it works not only when a, b, and c are whole numbers but also for fractions, decimals, mixed numbers, negative numbers, and irrational numbers.

- tells how addition and multiplication are related. Students often confuse addition with multiplication. They get 6×7 mixed up with $6 + 7$, and they often have difficulties in later grades knowing which operation to use in solving a contextual problem (Lemaire et al. 1994). Because the DP is expressed with equations that use both addition and multiplication, it helps students understand the difference between these two operations.

Pre-K–grade 3 learning progression

Understanding multiplication does not begin in third grade. Foundational learning experiences should be an explicit focus in prekindergarten through second grade. One important developmental skill for understanding multiplication is to decompose numbers between 1 and 10 in different ways. Children should begin to learn this skill before kindergarten (NRC 2009). A second important skill is for children to use fluency with decompositions to efficiently add a one-digit number to a two-digit number. Below, we describe the five steps of our learning progression, using Maria as an example.

Step 1. Decomposing small numbers

Maria knows how to count to ten. She also understands the order of the numbers, for example, that eight comes after seven and that five comes before six. Now she will take the next step and begin to understand how larger numbers are made up of smaller numbers. Through practicing, counting, and reasoning, she gradually learns to decompose all numbers up to ten. She learns to record this knowledge in simple addition equations, such as $3 + 3 = 6$ and $8 + 2 = 10$ (see **Activity 1** on **p. 307** and the **online activity sheet**).

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Maria knows that $4 + 1 = 5$. She also knows that she has five people in her family. Two are adults, and three are children, so $2 + 3 = 5$. In this manner, she learns that different ways exist to decompose numbers.

Step 2. Fluent mental addition of a one- and a two-digit number

Maria can now easily do such problems as $4 + 5$, but what about such problems as $18 + 5$? For example, Maria has signed up to bring treats on the day that some older students will be visiting the class to help with the math lesson. The class has eighteen students, and five people will visit. How many people should Maria plan for? She knows that 5 decomposes into 2 and 3. So she starts with 18, then adds 2 more people to take it up to 20, and then 3 more makes 23.

This same strategy helps Maria easily remember addition facts in a meaningful way: $9 + 8 = 17$ because it takes 1 more after 9 to get to 10, and then 7 more makes 17. She can also skip count starting from different numbers. She likes to count by nines because she just goes up ten and then back down one. So, if she starts at 27, the next one is 36, and then 45, and so forth. Eights are a little harder, but she can still go up ten and down two each time. Sixes and sevens are hardest for Maria. When her class plays the addition card game (see **activity 2** on **p. 307** and the **online activity sheet**), Maria has developed a strategy for dealing with sixes and sevens. She puts down a 6 or a 7 only when the last number ends in a 0, a 1, or a 2. This strategy is fine for now, but as Maria practices and gains experience, even the sixes and sevens will start to seem easy.

Step 3. Connecting area and addition

Maria likes to draw pictures on grid paper. She finds that the grid lines help her draw the

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> **After learning equal groups and arrays, the next step in understanding multiplication is using the distributive property.**

straight edges of houses, doors, and windows. Even though people's faces do not have straight lines, Maria likes to draw faces using the grid because there are so many ways to do it, and they always look funny.

Maria learns to count the number of unit squares in a shape by decomposing it into smaller shapes and adding (see **activity 3**). She likes to draw complicated shapes and then find ways to decompose the shapes to make counting the boxes easier.

Step 4. Connecting area and multiplication

Maria learns that rectangles are shapes that are easy to draw and understand on grid paper. With a rectangle, she can easily use skip counting to find the number of unit squares. For example, Maria draws a 3 × 5 rectangle on the day that it is her turn to put an array up on the class chart (see **activity 4**). She counts "5, 10, 15." Her friend Annie draws the same array but counts each square individually instead of skip counting. The answer is still 15! By making arrays and comparing solutions, the girls begin to understand one of three important meanings of multiplication: the area of a rectangle (see the **online CCSSM appendix**).

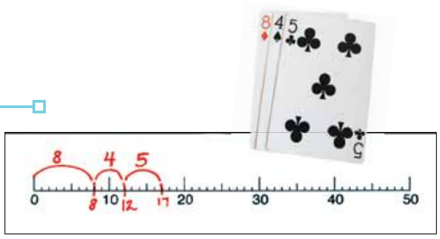
Step 5. Use the area model and the DP to make connections between multiplication problems

The first multiplication fact that challenged Maria was 6 × 4 = 24. One day, she is drawing rectangles and using multiplication to count the squares when she realizes that 6 × 4 is just 5 × 4 with one more row of 4! She already knows all her five facts, so now she remembers that 6 × 4 is the same as 20 + 4. On another day, Maria's teacher asks her eighteen students to figure out how many colored markers are in the classroom. Each student has a pack of six colored markers. Maria likes to use grid paper, so she decides to draw an 18 × 6 rectangle to figure out how many markers are in the class. Then she sees that she can decompose the 18 × 6 rectangle into a 10 × 6 rectangle and an 8 × 6 rectangle. She knows that 10 × 6 = 60 and that 8 × 6 = 48. So she adds 60 and 48, and finds that 108 markers are in the classroom.

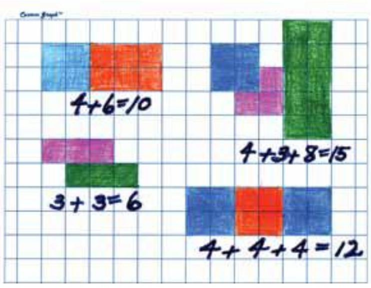
Illustrating the steps

Our five learning activities illustrate the steps of the learning progression. The activities are not single lessons; they are games and drawing activities that can be done multiple times to build and deepen understanding. Many variations are possible.

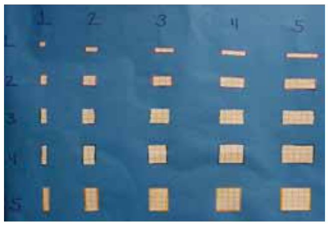
Activity 1 is about decomposing small numbers. Students practice thinking of ways to decompose numbers by playing a game where they have to put down two numbers that add up to a third. This decomposing leads naturally into the understanding of simple addition facts.

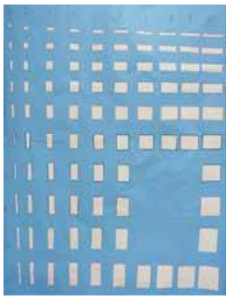


Activity 2 is about developing strategies to mentally add a one-digit number to a two-digit number. Again, this is practiced in the context of a game. This time it is a cooperative game, where a group of students help one another keep track of the numbers as the sum grows larger and larger.

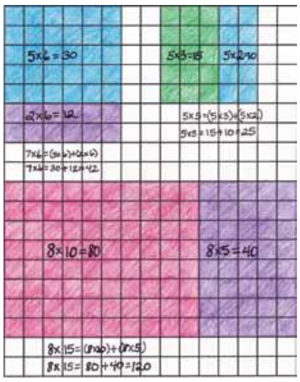


Activity 3 is about connecting area to addition. Students use grid paper to show how the area of a larger shape may be obtained as the sum of areas of two smaller shapes. We have observed that third graders who try to learn the area model of multiplication may be confused if they do not first understand that "area is additive" (3.MD.7.d). They may make the mistake of multiplying instead of adding when combining the areas of two rectangles.





Activity 4 is about understanding multiplication through the areas of rectangles. Students create arrays to illustrate one-digit multiplication facts. At this point, encourage students to think of a 2 × 3 array as *the meaning* of 2 × 3. The chart is not just a memory aid—it really shows what multiplication *is*. Even students who are still struggling with addition and skip counting can understand this meaning of multiplication as they make an array and count the unit squares. This activity can be done gradually over weeks or months, adding one or two arrays to the chart every day.



Activity 5 gives students a visual and computational tool for decomposing a more complicated multiplication problem into two easier ones. In doing this, students are in fact "using the area model to represent the distributive property" (3.MD.7.c). This approach can be used both as a way to learn the harder multiplication facts and as a way to develop strategies that lead to understanding the standard algorithm for multiplying multidigit numbers. Activities 4 and 5 teach students that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Students should be accustomed to thinking of zero as a number. Including it also makes the games more fun and interesting.

multiplication is a sense-making process rather than just memorization (Baroody 2006).

The Common Core State Standards for Mathematics document shows that the distributive property is in the forefront of student understanding of multiplication, beginning in the early grades. The DP helps students understand what multiplication means, how to break down complicated problems into simpler ones, and how to relate multiplication to area by using array models. The DP is a tool for reasoning about multiplication and applying it to other situations. Our sequence of activities illustrates how children might develop understanding from early concepts of decomposing numbers, up through addition, and, finally, in multiplication.

A note about zero

In the first two activities, face cards have the value of 0. We believe that it is important for students to get used to the fact that zero is a number. Having a zero also makes the games more fun and interesting. Students sometimes think it is funny that they can write $0 + 0 = 0$ as an addition sentence. In the game Adding Up the Deck (see **Activity 2**), students will often realize that when they are at a difficult number, such as 98, where they have to add past 100, they can make it easy by playing a 0 card. (Of course, this action just passes the problem to the next player. However, this is one aspect of the game that makes it engaging.)

The order of operations and parentheses

We have always used parentheses in equations where both addition and multiplication are present. Sometimes we have used parentheses for extra clarity, even when they are not strictly necessary. In dealing with equations that have more than one operation, students must begin to understand that order matters. Although parentheses are not officially introduced in CCSSM until grade 5, and order of operations is formally studied in grade 6, students should take beginning steps with these ideas in grade 3 (see the footnote in CCSSM 3.OA.8). For example, in the equation $3 + 2 \times 4 = 11$, students should understand to perform 2×4 first. Otherwise, they may add $3 + 2$ first and get 20 instead of 11. Using parentheses makes the equation clearer: $3 + (2 \times 4) = 11$ emphasizes that 2×4 is done first.

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Common Core Connections

3.OA.B.5
3.MD.C.7C
3.MD.7D

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1. Go fish
2. Adding up the deck
3. Making shapes on grid paper
4. Representing the whole multiplication table with arrays
5. Using the distributive property to understand multiplication