Math Task 1: Student-Invented Multiplication Strategies
The following task is a modification of a task taken from "Elementary and Middle School Mathematics: Teaching Developmentally", 8e, John A. Van de Walle.

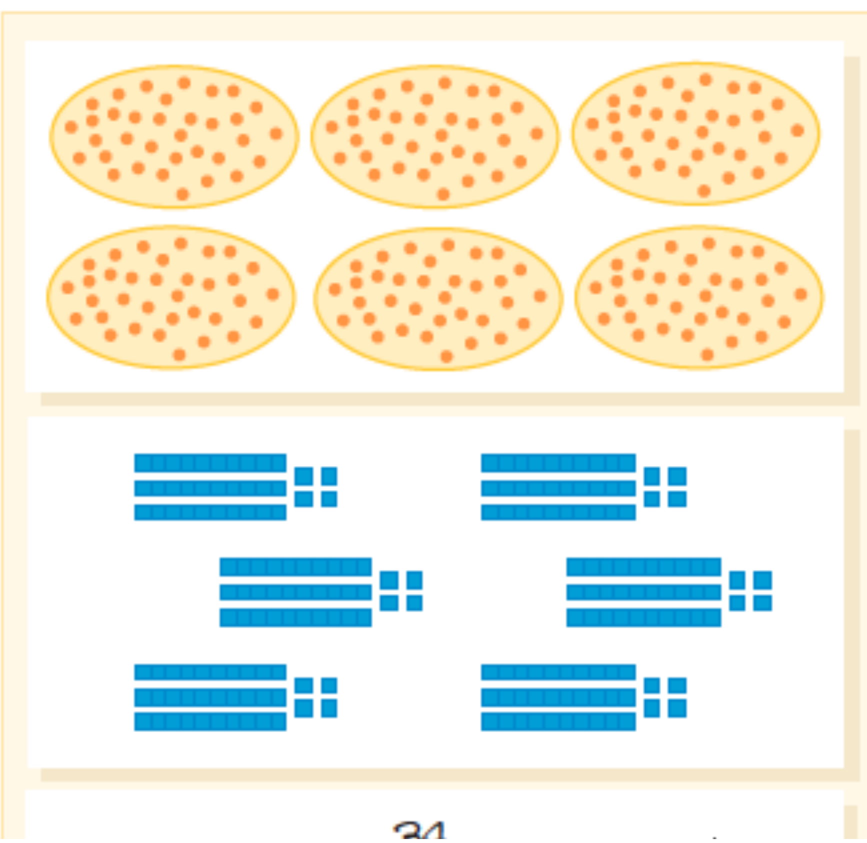
Read the following exert on the proceeding pages from "Elementary and Middle School Mathematics: Teaching Developmentally", 8e, John A. Van de Walle about complete-number, partitioning, and compensation strategies for multiplication.

Answer the question "How many legs do 29 elephants have?" using each of the three strategies listed, so you will have three different complete solutions to the problem. Give an advantage of each method.

Multiplication by a Single-Digit Multiplier

As with addition and subtraction, it is helpful to place multiplication tasks in context. (Be sure students, in particular ELLs, understand the context that is selected.) Let students model the problems in ways that make sense to them. Do not be concerned about reversing factors (6 sets of 34 or 34 sets of 6). Nor should you be timid about the numbers you use. The problem 3×24 may be easier than 7×65, but the latter provides a challenge. The types of strategies that students use for multiplication are much more varied than for addition and subtraction. The three categories described here are strategies grounded in student reasoning, as described in research on multiplicative reasoning (Baek, 2006; Confrey, 2008; Petit, 2009).

Complete-Number Strategies (Including Doubling). Students who are not yet comfortable breaking numbers into parts will approach the numbers in the sets as single groups. Most likely, these early strategies will be based on repeated addition. Often students will list long columns of numbers and add them up. In an attempt to shorten this process, students soon realize that if they add two numbers, the next two



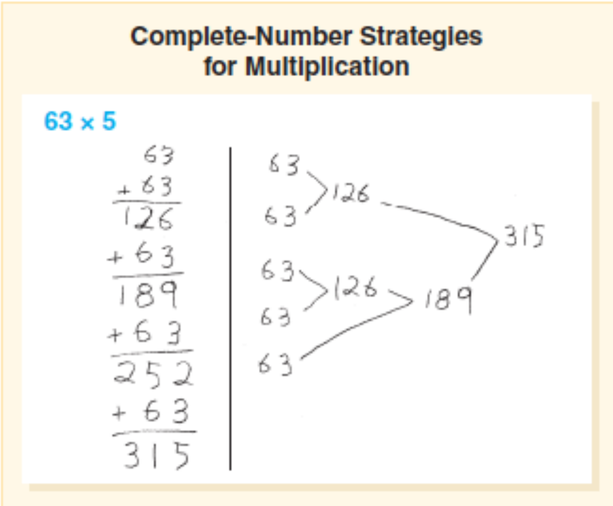


FIGURE 13.2 Students who use a complete-number strategy do not break numbers apart into decades or tens and ones.

will have the same sum and so on down the line. This doubling can become the principal approach for many students (Flowers & Rubenstein, 2010/2011). Doubling capitalizes on the distributive property, whereby doubling 47 is double 40 + double 7, and the associative property, in which doubling 7 tens—or 2 × (7 × 10)—is the same as 10 times double 7 or (2 × 7) × 10. Figure 13.2 illustrates two methods students may use. Students who use these strategies will benefit from listening to other students who use base-ten models. They may also need more work with base-ten grouping activities where they take numbers apart in different ways.

Partitioning Strategies. Students break numbers up in a variety of ways that reflect an understanding of base-ten concepts, at least four of which are illustrated in Figure 13.3. The By Decades partitioning strategy (which can be extended to By Hundreds, By Thousands, etc.) is the same as the standard algorithm except that students always begin with the largest values. It extends easily to three digits and is very powerful as a mental strategy. Another valuable strategy for mental methods is to compute mentally with multiples of 25 and 50 and then add or subtract a small adjustment. All partition strategies rely on knowledge of the distributive property.

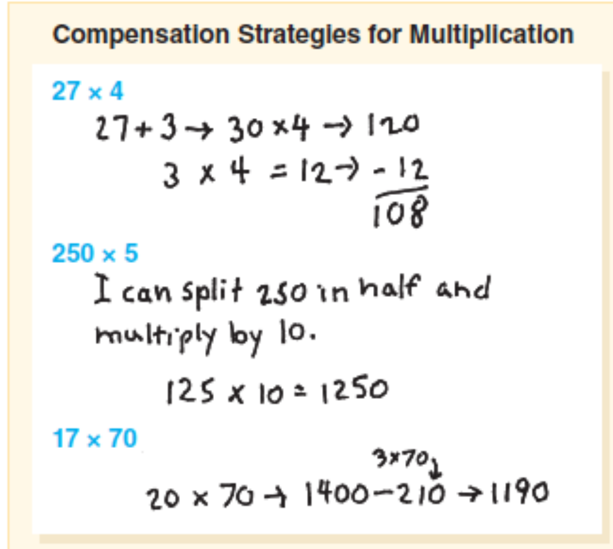
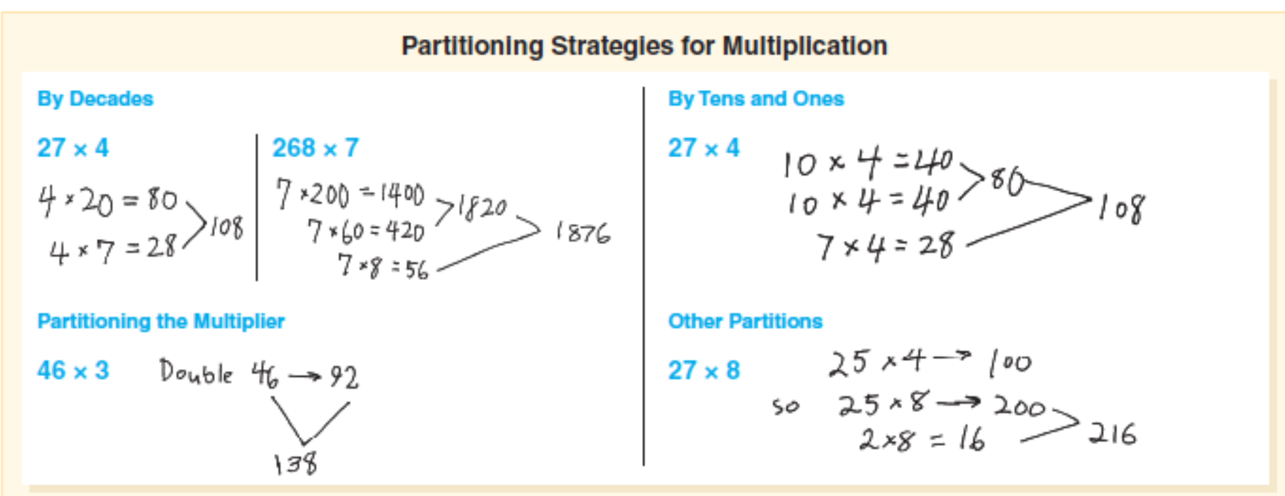


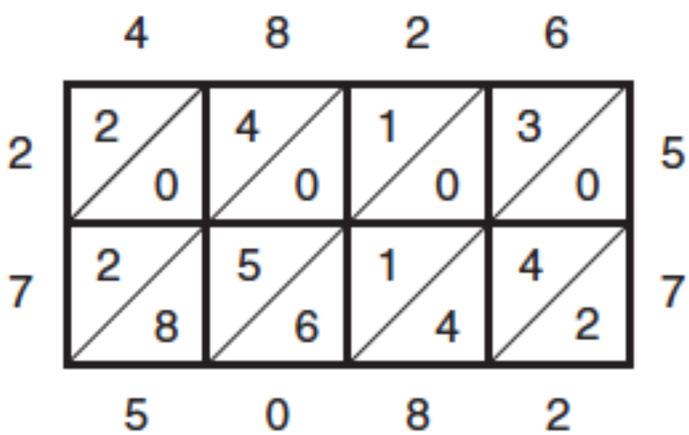
FIGURE 13.4 Compensation methods use a product related to the original. A compensation is made in the answer, or one factor is changed to compensate for a change in the other factor.

Compensation Strategies. Students and adults look for ways to manipulate numbers so that the calculations are easy. In Figure 13.4, the problem 27 × 4 is changed to an easier one, and then an adjustment or compensation is made. In the second example, one factor is cut in half and the other doubled. This is often used when a 5 or a 50 is involved. Because these strategies are so dependent on the numbers involved, they can't be used for all computations. However, they are powerful strategies, especially for mental math and estimation.



Math Task 2: Multiplication Algorithms

One of the popular schemes used for multiplying in the fifteenth century was called the **lattice method.** The two numbers to be multiplied, 4826 and 57 in the example shown here, are written above and to the right of the lattice. The partial products are written in the cells. The sums of numbers along the diagonal cells, beginning at the lower right with 2, 4 + 4 + 0, etc., form the product 275,082.



1. Use lattice multiplication to multiply 36 x 17.
2. Use a traditional algorithm to multiply 36 x 17.
3. What is similar in the two methods (use appropriate terminology, ie. place value ect)?
4. What is different about the two methods? What are advantages of each?

Math Task 3: Division Algorithms

Read the following exert on the proceeding pages from "Elementary and Middle School Mathematics: Teaching Developmentally", 8e, John A. Van de Walle about the traditional long division algorithm with one-digit divisors. Then complete the following problems.

Complete the division problem 136 ÷ 4 using the following methods.

a) Using the terminology on the previous pages, complete the above long division problems explaining every step of the process. Also neatly show your long division algorithm. Give an advantage of this method.

b) Use base ten pieces to illustrate the above long division algorithm for the problem above. Show what base-ten pieces correspond to each digit in the quotient (there is an example of this in your competency quiz practice problems). Give an advantage of this method.

c) Use a rectangular array of base-ten pieces to illustrate the quotient for the problem above (there is an example of this in your competency quiz practice problems). Give an advantage of this method.

Notice that the missing-factor strategy works equally well for one-digit divisors as for two-digit divisors. Also notice that it is okay to include division problems in the cluster. In the first example, 400 ÷ 4 could easily have replaced 100 × 4, and 125 × 4 could replace 500 ÷ 4. The idea is to capitalize on the inverse relationship between multiplication and division.

Cluster problems provide students with a sense that problems can be solved in different ways and with different starting points. Therefore, rather than cluster problems, you can provide students with a variety of first steps for solving a problem. Their task is to select one of the starting points and solve the problem from there. For example, here are four possible starting points for 514 ÷ 8:

10 × 8 400 ÷ 8 60 × 8 80 ÷ 8

When students are first asked by the teacher to solve problems using two methods, they often use a primitive or completely inefficient method for their second approach (or revert to a standard algorithm). For example, to solve 514 ÷ 8, a student might perform a very long string of repeated subtractions (514 – 8 = 506, 506 – 8 = 498, 498 – 8 = 490, and so on) and count how many times he or she subtracted 8. Others will actually draw 514 tally marks and loop groups of 8. These students have not developed sufficient flexibility to think of other efficient methods. The idea just suggested of posing a variety of starting points can nudge students into other more profitable alternatives. Class discussions will also help students begin to see more flexible approaches.

Standard Algorithm for Division

Long division is the one standard algorithm that starts with the left-hand or biggest pieces. The conceptual basis for the algorithm most often taught in textbooks is the partition or fair-share method, the method we will explore in detail here. Another well-known algorithm is based on repeated subtraction and may be viewed as a good way to record the missing-factor approach with partial products recorded in a column to the right of the division computation. This may be the preferred strategy for some students, especially students from other countries and students with learning disabilities. Keep in mind that students from different countries learn other "standard" algorithms, and these should be valued and shared with other students as successful options for doing division. As shown by the two examples in Figure 13.11, one advantage of the standard algorithm that's used in the United States is that there is total flexibility in the factors selected at each step of the way. This is important for students who struggle, as they can select facts they know and work from that point.

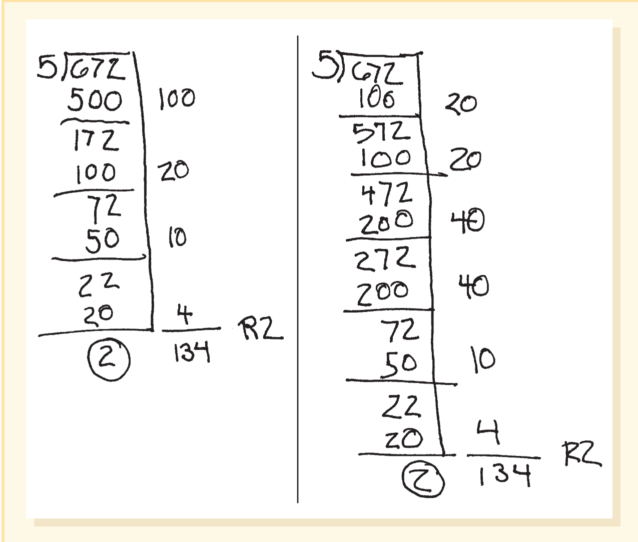


FIGURE 13.11 In the division algorithm shown, the numbers on the side indicate the quantity of the divisor being subtracted from the dividend. As the two examples indicate, the divisor can be subtracted repeatedly from the dividend in groups of any amount.

One-Digit Divisors

Typically, the division algorithm with one-digit divisors is introduced in the third grade, and it should provide the basis for two-digit divisors. Students in the upper grades who are struggling with the division algorithm can also benefit from a conceptual development.

Begin with Models. Traditionally, if we were to do a problem such as $4\overline{)583}$, we might say, "4 goes into 5 one time." This is quite mysterious to students. How can you just ignore the "83" and keep changing the problem? Preferably, you want students to think of the 583 as 5 hundreds, 8 tens, and 3 ones, not as the independent digits 5, 8, and 3. One idea is to use a context such as candy bundled in boxes of ten with 10 boxes (100) to a carton. Then the problem becomes as follows: *We have 5 cartons, 8 boxes, and 3 pieces of candy to share evenly between 4 schools.* In this context, it is reasonable to share the cartons first until no more can be shared. Those remaining are "unpacked," and the boxes shared, and so on. Money ($100, $10, and $1) can be used in a similar manner.

PAUSE and REFLECT

Try this yourself using base-ten pieces, four paper plates (or pieces of paper), and the problem 583 ÷ 4. Try to talk through the process without using "goes into." Think sharing.

Language plays an enormous role in thinking about the standard algorithm conceptually. Most adults are so accustomed to the "goes into" language that it is hard to let it go. For the problem 583 ÷ 4, here is some suggested language:

- I want to share 5 hundreds, 8 tens, and 3 ones among these four sets. There are enough hundreds for each set to get 1 hundred. That leaves 1 hundred that I can't share.
- I'll trade the hundred for 10 tens. That gives me a total of 18 tens. I can give each set 4 tens and have 2 tens left over. Two tens are not enough to go around the four sets.
- I can trade the 2 tens for 20 ones and put those with the 3 ones I already had. That makes a total of 23 ones. I can give 5 ones in each of the four sets. That leaves me with 3 ones as a remainder. In all, I gave out to each group 1 hundred, 4 tens, and 5 ones with 3 left over.

Develop the Written Record. The recording scheme for the long-division algorithm is not completely intuitive. You will need to be quite directive in helping students learn to record the fair sharing with models. There are essentially four steps:

1. *Share* and record the number of pieces put in each group.
2. *Record* the number of pieces shared in all. Multiply to find this number.
3. *Record* the number of pieces remaining. Subtract to find this number.
4. *Trade* (if necessary) for smaller pieces, and combine with any that are there already. Record the new total number in the next column.

When students model problems with a one-digit divisor, steps 2 and 3 seem unnecessary. Explain that these steps really help when you don't have the pieces there to count.

Record Explicit Trades. Figure 13.12 details each step of the recording process just described. On the left, you see the standard algorithm. To the right is a suggestion that matches the actual action with the models by explicitly recording the trades. Instead of the somewhat mysterious "bring-down" step of the standard algorithm, the traded pieces are crossed out, as is the number of existing pieces in the next column. The combined number of pieces is written in this column using a two-digit number. In the example, 2 hundreds are traded for 20 tens, combined with the 6 that were there for a total of 26 tens. The 26 is, therefore, written in the tens column.

Students who are required to make sense of the long-division procedure find this *explicit-trade* method easier to follow. (*Author note:* The explicit-trade method is an invention of John Van de Walle. It has been used successfully in grades 3 to 8. You will not find it in other textbooks.) Blank division charts with wide place-value columns are highly recommended for this method. Such charts can be found in Blackline Master 20. Without the charts, it is important to spread out the digits in the dividend when writing down the problem.

Both the explicit-trade method and the use of place-value columns will help with the problem of leaving out a middle zero in a problem (see Figure 13.13).

Two-Digit Divisors

The *Common Core State Standards* states that fifth-grade students should be able to find "whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division." The CCSSO goes on to state that the student should "[i]llustrate and explain the calculation by using equations, rectangular arrays, and/or area models" (p. 35). In the past, a large chunk of the fourth, fifth, and sometimes sixth grade was spent on this skill (long division), with the results often being that students acquire negative attitudes toward mathematics and many students not mastering the skill.

An Intuitive Idea. Suppose that you were sharing a large pile of candies with 36 friends. Instead of passing them out one at a time, you conservatively estimate that each person could get at least 6 pieces. So you give 6 to each of your friends. Now you find there are more than 36 pieces left. Do you have everyone give back the 6 pieces so you can then give them 7 or 8? That would be silly! You simply pass out more.

The candy example gives us two good ideas for sharing in long division. First, always underestimate how much can be shared. You can always pass out some more. To avoid ever overestimating, always pretend there are more sets among which to share than there really are. For example, if you are dividing 312 by 43 (sharing among 43 sets or "friends"), pretend you have 50 sets instead. Round *up* to the next multiple of 10. You can easily determine that 6 pieces can be shared among 50 sets because 6×50 is an easy product. Therefore, since there are really only 43 sets, clearly you can give *at least* 6 to each. Always consider a larger divisor; *always round up.*

Using the Idea Symbolically. These ideas are used in Figure 13.14. Both the standard algorithm and the explicit-trade method of recording are illustrated. The rounded-up divisor, 70, is written in a little "think bubble" above the real divisor. Rounding up has another advantage: It is

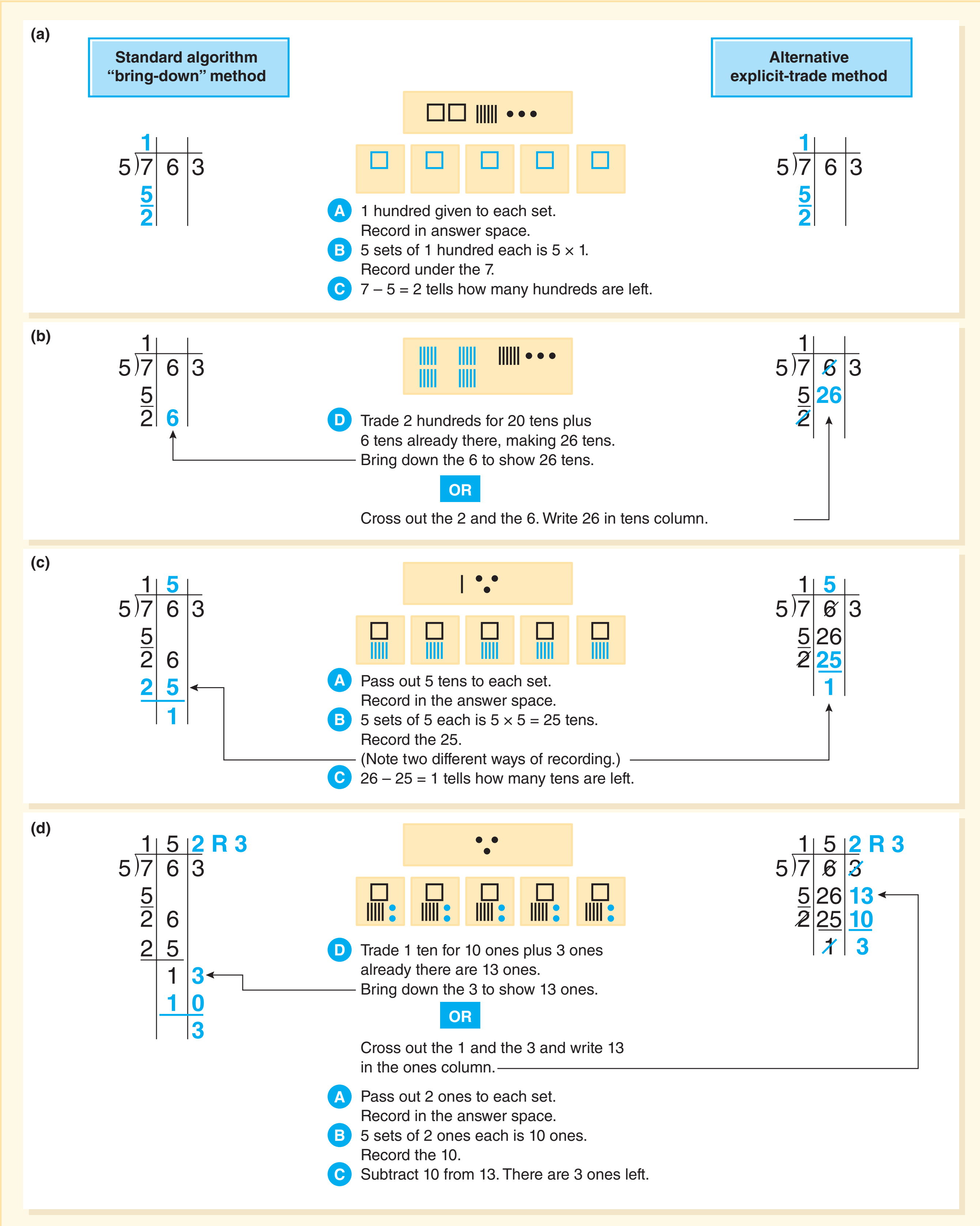


FIGURE 13.12 The standard algorithm and explicit-trade methods are connected to each step of the division process. Every step can and should make sense (see Blackline Master 20).