Section 3.3 MULTIPLICATION

Corning Tower at the Empire State Plaza, Albany, New York

PROBLEM OPENER

Lee has written a two-digit number in which the units digit is her favorite digit. When she subtracts the tens digit from the units digit, she gets 3. When she multiplies the original two-digit number by 21, she gets a three-digit number whose hundreds digit is her favorite digit and whose tens and units digits are the same as those in her original two-digit number. What is her favorite digit?

The skyscraper in the photo here is called the Corning Tower. A window-washing machine mounted on top of the building lowers a cage on a vertical track so that each column of 40 windows can be washed. After one vertical column of windows has been washed, the machine moves to the next column. The rectangular face of the building visible in the photograph has 36 columns of windows. The total number of windows is $40 + 40 + 40 + \ldots + 40$, a sum in which 40 occurs 36 times. This sum equals the product 36×40, or 1440. We are led to different expressions for the sum and product by considering the rows of windows across the floors. There are 36 windows in each floor on this face of the building and 40 floors. Therefore, the number of windows is $36 + 36 + 36 + \ldots + 36$, a sum in which 36 occurs 40 times. This sum is equal to 40×36, which is also 1440. For sums such as these in which one number is repeated, multiplication is a convenient method for doing addition.

Historically, multiplication was developed to replace certain special cases of addition, namely, the cases of *several equal addends.* For this reason we usually see **multiplication** of whole numbers explained and defined as **repeated addition.**

Multiplication of Whole Numbers For any whole numbers r and s, the product of r and s is the sum with s occurring r times. This is written as

$$r \times s = \underbrace{s + s + s + \ldots + s}_{r \text{ times}}$$

If $r \neq 0$ and $s \neq 0$, r and s are called **factors.**

One way of representing multiplication of whole numbers is with a **rectangular array** of objects, such as the rows and columns of windows at the beginning of this section. Figure 3.12 shows the close relationship between the use of *repeated addition* and *rectangular arrays* for illustrating products. Part (a) of the figure shows squares in 4 groups of 7 to illustrate 7 + 7 + 7 + 7, and part (b) shows the squares pushed together to form a 4×7 rectangle.

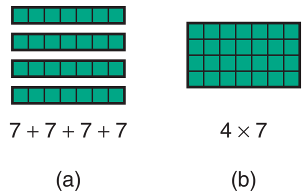


Figure 3.12

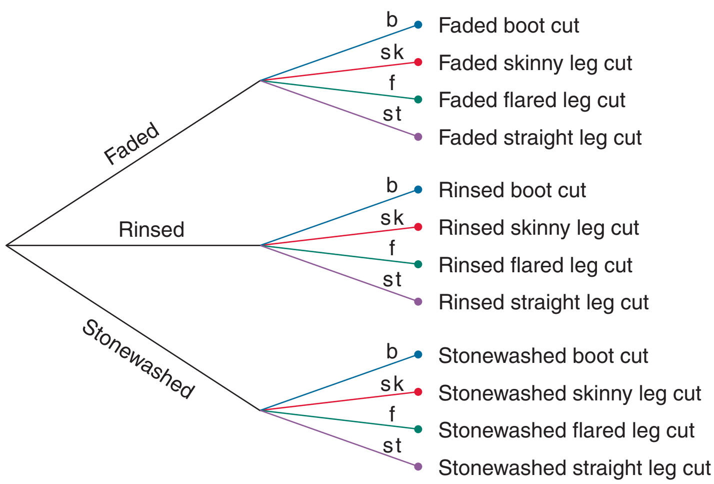
In general, $r \times s$ is the number of objects in an $r \times s$ rectangular array.

Another way of viewing multiplication is with a figure called a **tree diagram.** Constructing a tree diagram is a counting technique that is useful for certain types of multiplication problems.

EXAMPLE A

A catalog shows jeans available in faded, rinsed, or stonewashed fabric and in boot (b), skinny leg (sk), flared leg (f), or straight leg (st) cuts. How many types of jeans are available?

Solution A tree diagram for this problem is shown below. The tree begins with 3 branches, each labeled with one of the types of fabric. Each of these branches leads to 4 more branches, which correspond to the cuts. The tree has $3 \times 4 = 12$ endpoints, one for each of the 12 different types of jeans.



NCTM Standards

Research provides evidence that students will rely on their own computational strategies (Cobb et al.). Such inventions contribute to their mathematical development (Gravemeijer; Steffe). p. 86

MODELS FOR MULTIPLICATION ALGORITHMS

Physical models for multiplication can generate an understanding of multiplication and suggest or motivate procedures and rules for computing. There are many suitable models for illustrating multiplication. Base-ten pieces are used in the following examples.

Figure 3.13 illustrates 3×145, using base-ten pieces. First 145 is represented as shown in (a). Then the base-ten pieces for 145 are tripled. The result is 3 flats, 12 longs, and 15 units, as shown in (b). Finally, the pieces are regrouped: 10 units are replaced by 1 long, leaving 5 units; and 10 longs are replaced by 1 flat, leaving 3 longs. The result is 4 flats, 3 longs, and 5 units, as shown in (c).

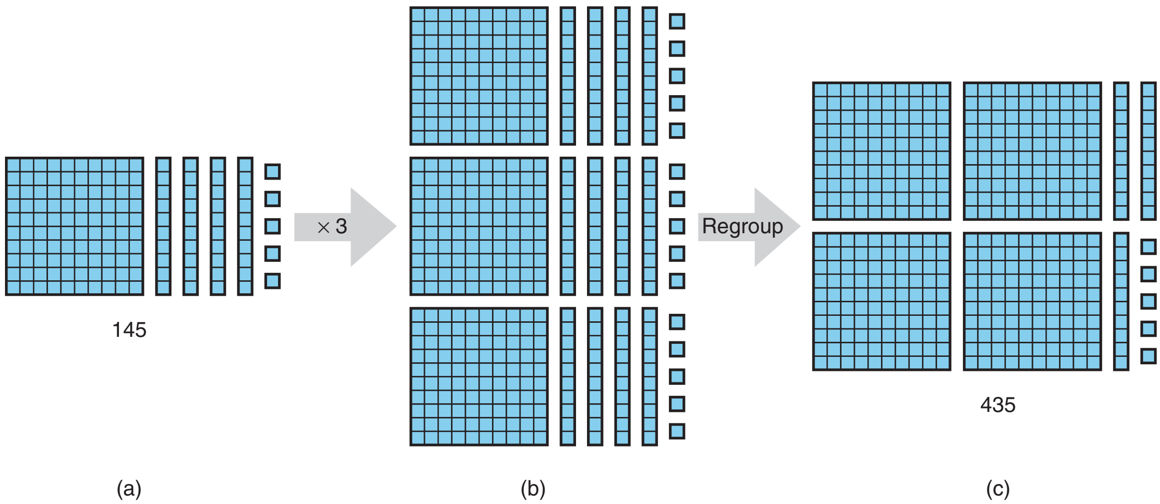


Figure 3.13

Base-ten pieces can be used to illustrate the pencil-and-paper algorithm for computing. Consider the product 3×145 shown in Figure 3.13. First a 5, indicating the remaining 5 units in part (c), is recorded in the units column, and the 10 units that have been regrouped are recorded by writing 1 in the tens column (see below). Then 3 is written in the tens column for the remaining 3 longs, and 1 is recorded in the hundreds column for the 10 longs that have been regrouped.

Flats	Longs	Units
1	1	
1	4	5
	×	3
4	3	5

Figure 3.14 on page 167 illustrates how multiplication by 10 can be carried out with base-ten pieces. Multiplying by 10 is especially convenient because 10 units can be placed together to form 1 long, 10 longs to form 1 flat, and 10 flats to form 1 long-flat (row of flats).

To multiply 34 and 10, we replace each base-ten piece for 34 by the base-ten piece for the next higher power of 10 (see Figure 3.14). We begin with 3 longs and 4 units and end with 3 flats, 4 longs, and 0 units. This illustrates the familiar fact that the product of any whole number and 10 can be computed by placing a zero at the right end of the numeral for the whole number.

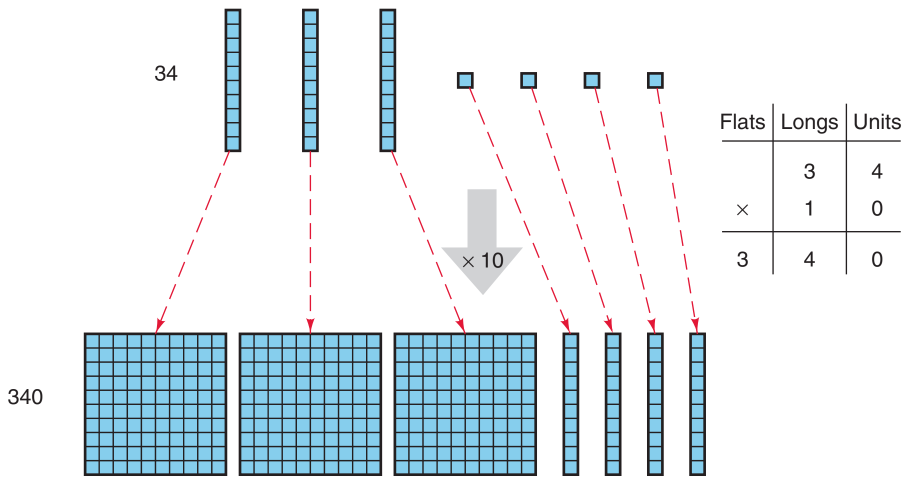


Figure 3.14

Computing the product of two numbers by repeated addition of base-ten pieces becomes impractical as the size of the numbers increases. For example, computing 18 × 23 requires representing 23 with base-ten pieces 18 times. For products involving two-digit numbers, rectangular arrays are more convenient.

To compute 18 × 23, we can draw a rectangle with dimensions 18 by 23 on grid paper (Figure 3.15). The product is the number of small squares in the rectangular array. This number can be easily determined by counting groups of 100 flats and strips of 10 longs. The total number of small squares is 414. Notice how the array in Figure 3.15 can be viewed as 18 horizontal rows of 23, once again showing the connection between the repeated-addition and rectangular-array views of multiplication.

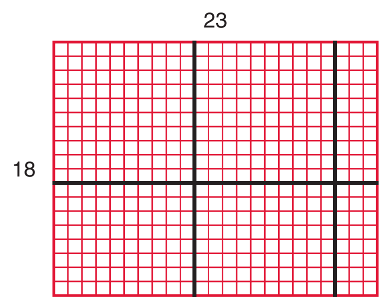


Figure 3.15

The pencil-and-paper algorithm for multiplication computes **partial products.** When a two-digit number is multiplied by a two-digit number, there are four partial products.

The product 13 × 17 is illustrated in Figure 3.16. The four regions of the grid formed by the heavy lines represent the four partial products. Sometimes it is instructive to draw arrows from each partial product to the corresponding region on the grid.

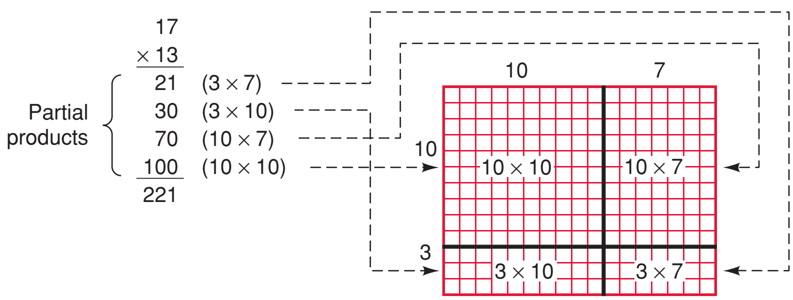


Figure 3.16

HISTORICAL HIGHLIGHT

$\rightarrow 1 \times 52 = 52$
$\rightarrow 2 \times 52 = 104$
$4 \times 52 = 208$
$\rightarrow 8 \times 52 = 416$

$$\begin{array}{r} 52 \\ 104 \\ +416 \\ \hline 572 \end{array}$$

One of the earliest methods of multiplication is found in the Rhind Papyrus. This ancient scroll (ca. 1650 B.C.E.), more than 5 meters in length, was written to instruct Egyptian scribes in computing with whole numbers and fractions. Beginning with the words "Complete and thorough study of all things, insights into all that exists, knowledge of all secrets . . . ," it indicates the Egyptians' awe of mathematics. Although most of its 85 problems have a practical origin, there are some of a theoretical nature. The Egyptians' algorithm for multiplication was a succession of doubling operations, followed by addition as shown in the example at the left. To compute 11×52, they would repeatedly double 52, then add *one* 52, *two* 52s, and *eight* 52s to get *eleven* 52s.

NUMBER PROPERTIES

Four properties for addition of whole numbers were stated in Section 3.2. Four corresponding properties for multiplication of whole numbers are stated below, along with one additional property that relates the operations of addition and multiplication.

Closure Property for Multiplication This property states that the product of any two whole numbers is also a whole number.

> **Closure Property for Multiplication** For any whole numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,
>
> $\boldsymbol{a \times b}$ is a unique whole number.

Identity Property for Multiplication The number 1 is called an **identity for multiplication** because when multiplied by another number, it leaves the identity of the number unchanged. For example,

$$1 \times 14 = 14 \qquad 34 \times 1 = 34 \qquad 1 \times 0 = 0$$

The number 1 is unique in that it is the only number that is an identity for multiplication.

> **Identity Property for Multiplication** For any whole number $\boldsymbol{b}$,
>
> $$\mathbf{1} \times \boldsymbol{b} = \boldsymbol{b} \times \mathbf{1} = \boldsymbol{b}$$
>
> and 1 is the unique identity for multiplication.

Commutative Property for Multiplication This number property says in any product of numbers, two numbers may be interchanged (commuted) without affecting the product. This property is called the **commutative property for multiplication.** For example,

$$\underbrace{347 \times 26}_{\uparrow} = \underbrace{26 \times 347}_{\uparrow}$$

Commutative property
for multiplication

Commutative Property for Multiplication For any whole numbers ***a*** and ***b***,

$$a \times b = b \times a$$

NCTM Standards

Using area models, properties of operations such as commutativity of multiplication become more apparent. p. 152

The commutative property is illustrated in Figure 3.17, which shows two different views of the same rectangular array. Part (a) represents 7×5, and part (b) represents 5×7. Since part (b) is obtained by rotating part (a), both figures have the same number of small squares, so 7×5 is equal to 5×7.

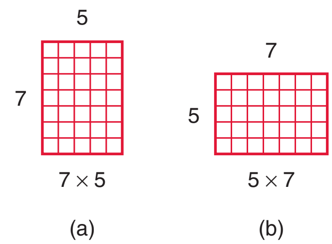


Figure 3.17

As the multiplication table in Figure 3.18 shows, the commutative property for multiplication approximately cuts in half the number of basic multiplication facts that elementary school students must memorize. Each product in the shaded part of the table corresponds to an equal product in the unshaded part of the table.

EXAMPLE B

Since $3 \times 7 = 21$, we know by the commutative property for multiplication that $7 \times 3 = 21$. What do you notice about the location of each product in the shaded part of the table relative to the location of the corresponding equal product in the unshaded part of the table?

Solution If the shaded part of the table is folded along the diagonal onto the unshaded part, each product in the shaded part will coincide with an equal product in the unshaded part. In other words, the table is symmetric about the diagonal from upper left to lower right.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure 3.18

Notice that the numbers in the rows of the multiplication table in Figure 3.18 form arithmetic sequences, for example, 2, 4, 6, 8, . . . and 3, 6, 9, 12. . . . One reason that children learn to count by 2s, 3s, and 5s is to acquire background for learning basic multiplication facts.

Associative Property for Multiplication In any product of three numbers, the middle number may be associated with and multiplied by either of the two end numbers. This property is called the **associative property for multiplication.** For example,

$$\underbrace{6 \times (7 \times 4)} = \underbrace{(6 \times 7) \times 4}$$

Associative property
for multiplication

> **Associative Property for Multiplication** For any whole numbers a, b, and c,
>
> $$a \times (b \times c) = (a \times b) \times c$$

Figure 3.19 illustrates the associative property for multiplication. Part (a) represents 3×4, and (b) shows 5 of the 3×4 rectangles. The number of small squares in (b) is $5 \times (3 \times 4)$. Part (c) is obtained by subdividing the rectangle (b) into 4 copies of a 3×5 rectangle. The number of small squares in (c) is $4 \times (3 \times 5)$, which, by the commutative property for multiplication, equals $(5 \times 3) \times 4$. Since the numbers of small squares in (b) and (c) are equal, $5 \times (3 \times 4) = (5 \times 3) \times 4$.

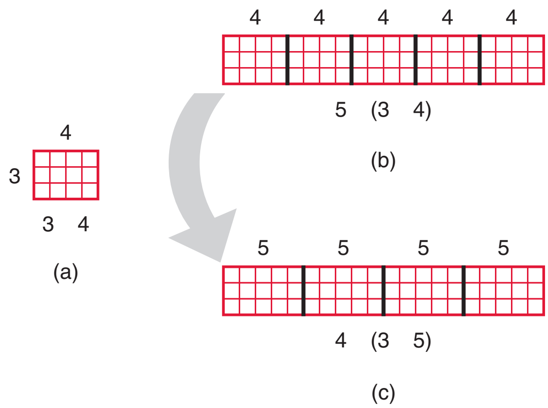


Figure 3.19

The commutative and associative properties are often used to obtain convenient combinations of numbers for mental calculations, as in Example C.

EXAMPLE C

Try computing $25 \times 46 \times 4$ in your head before reading further.

Solution The easy way to do this is by rearranging the numbers so that 25×4 is computed first and then 46×100. The following equations show how the commutative and associative properties permit this rearrangement.

Associative property
for multiplication

$$(25 \times 46) \times 4 = (46 \times 25) \times 4 = 46 \times (25 \times 4)$$

Commutative property
for multiplication

Distributive Property When multiplying a sum of two numbers by a third number, we can add the two numbers and then multiply by the third number, or we can multiply each number of the sum by the third number and then add the two products.

For example, to compute $35 \times (10 + 2)$, we can compute 35×12, or we can add 35×10 to 35×2. This property is called the **distributive property for multiplication over addition.**

$$35 \times 12 = \underbrace{35 \times (10 + 2)}_{\uparrow} = \underbrace{(35 \times 10) + (35 \times 2)}_{\uparrow}$$

Distributive property

> **Distributive Property for Multiplication over Addition** For any whole numbers ***a, b,*** and ***c,***
>
> $$\boldsymbol{a \times (b + c) = a \times b + a \times c}$$

One use of the distributive property is in learning the basic multiplication facts. Elementary schoolchildren are often taught the "doubles" ($2 + 2 = 4$, $3 + 3 = 6$, $4 + 4 = 8$, etc.) because these number facts together with the distributive property can be used to obtain other multiplication facts.

EXAMPLE D How can $7 \times 7 = 49$ and the distributive property be used to compute 7×8?

Solution

$$7 \times 8 = \underbrace{7 \times (7 + 1)}_{\uparrow} = \underbrace{49 + 7}_{\uparrow} = 56$$

Distributive property

The distributive property can be illustrated by using rectangular arrays, as in Figure 3.20. The dimensions of the array in (a) are 6 by $(3 + 4)$, and the array contains 42 small squares. Part (b) shows the same squares separated into two rectangular arrays with dimensions 6 by 3 and 6 by 4. Since the number of squares in both figures is the same, $6 \times (3 + 4) = (6 \times 3) + (6 \times 4)$.

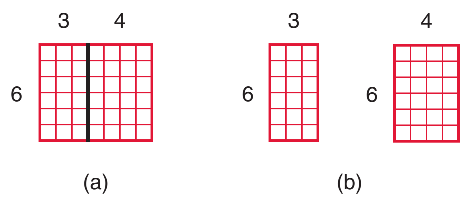


Figure 3.20

The distributive property also holds for multiplication over subtraction.

Explore

Math Activity for 3-2
Multiply Mentally

MAIN IDEA

I will mentally multiply a one-digit factor by a two-digit factor.

Math Online

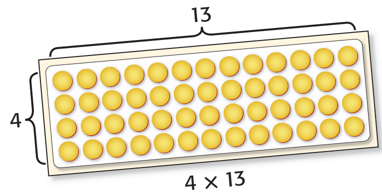
macmillanmh.com

• Concepts in Motion

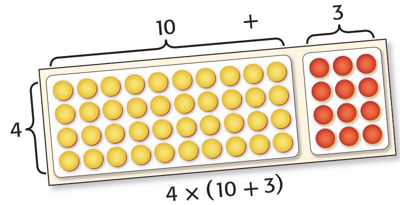
It may be hard to find a product like 4 × 13 mentally, even if you use counters. If you separate the counters into smaller groups called *partial products*, it is easier to multiply.

ACTIVITY

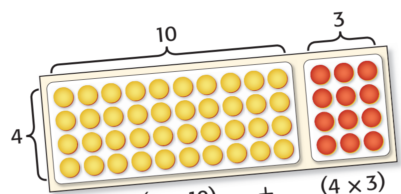
1 Find 4 × 13 mentally using partial products.



Model 4 × 13 by arranging counters in 4 rows and 13 columns.



Separate 13 into two numbers that are each easily multiplied by 4.



Multiply to find the number of counters in each group. Then add.

(4 × 10) + (4 × 3)
40 + 12 = 52

Rewrite 4 × 13 as (4 × 10) + (4 × 3). This is useful since it is easier to find (4 × 10) + (4 × 3) mentally than to find 4 × 13. So, 4 × 13 is 52.

Think About It

1. To find 4 × 13, you can also find 4 × (9 + 4). Why is it easier to find 4 × (10 + 3) mentally than it is to find 4 × (9 + 4)?

2. Which expression would you use to find 7 × 19 mentally: 7 × (13 + 6) or 7 × (10 + 9)? Explain.

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EXAMPLE E

Show that the two sides of the following equation are equal.

$$6 \times (20 - 8) = (6 \times 20) - (6 \times 8)$$

Solution $6 \times (20 - 8) = 6 \times 12 = 72$ and $(6 \times 20) - (6 \times 8) = 120 - 48 = 72$.

MENTAL CALCULATIONS

In the following paragraphs, three methods are discussed for performing mental calculations of products. These methods parallel those used for performing mental calculations of sums and differences.

Compatible Numbers for Mental Calculation We saw in Example C that the commutative and associative properties permit the rearrangement of numbers in products. Such rearrangements can often enable computations with compatible numbers.

EXAMPLE F

Find a more convenient arrangement that will yield compatible numbers, and compute the following products mentally.

1. $5 \times 346 \times 2$
2. $2 \times 25 \times 79 \times 2$

Solution **1.** $5 \times 2 \times 346 = 10 \times 346 = 3460$. **2.** $2 \times 2 \times 25 \times 79 = 100 \times 79 = 7900$.

Substitutions for Mental Calculation In certain situations the distributive property is useful for facilitating mental calculations. For example, to compute 21×103, first replace 103 by $100 + 3$ and then compute 21×100 and 21×3 in your head. Try it.

$$21 \times 103 = \underbrace{21 \times (100 + 3)} = \underbrace{2100 + 63} = 2163$$

Distributive property

Occasionally it is convenient to replace a number by the difference of two numbers and to use the fact that multiplication distributes over subtraction. Rather than compute 45×98, we can compute 45×100 and subtract 45×2.

$$45 \times 98 = \underbrace{45 \times (100 - 2)} = \underbrace{4500 - 90} = 4410$$

Distributive property

EXAMPLE G

Find a convenient substitution, and compute the following products mentally.

1. 25×99
2. 42×11
3. 34×102

Solution **1.** $25 \times (100 - 1) = 2500 - 25 = 2475$. **2.** $42 \times (10 + 1) = 420 + 42 = 462$. **3.** $34 \times (100 + 2) = 3400 + 68 = 3468$.

NCTM Standards

Other relationships can be seen by decomposing and composing area models. For example, a model for 20×6 can be split in half and the halves rearranged to form a 10×12 rectangle, showing the equivalence of 10×12 and 20×6. p. 152

Equal Products for Mental Calculation This method of performing mental calculations is similar to the *equal differences* method used for subtraction. It is based on the fact that the product of two numbers is unchanged when one of the numbers is divided by a given number and the other number is multiplied by the same number. For example, the product 12×52 can be replaced by 6×104 by dividing 12 by 2 and multiplying 52 by 2. At this point we can mentally calculate 6×104 to be 624. Or we can continue the process of dividing and multiplying by 2, replacing 6×104 by 3×208, which can also be mentally calculated.

Figure 3.21 illustrates why one number in a product can be halved and the other doubled without changing the product. The rectangular array in part (a) of the figure represents 22×16. If this rectangle is cut in half, the two pieces can be used to form an 11×32 rectangle, as in (b). Notice that 11 is half of 22 and 32 is twice 16. Since the rearrangement has not changed the number of small squares in the two rectangles, the products 22×16 and 11×32 are equal.

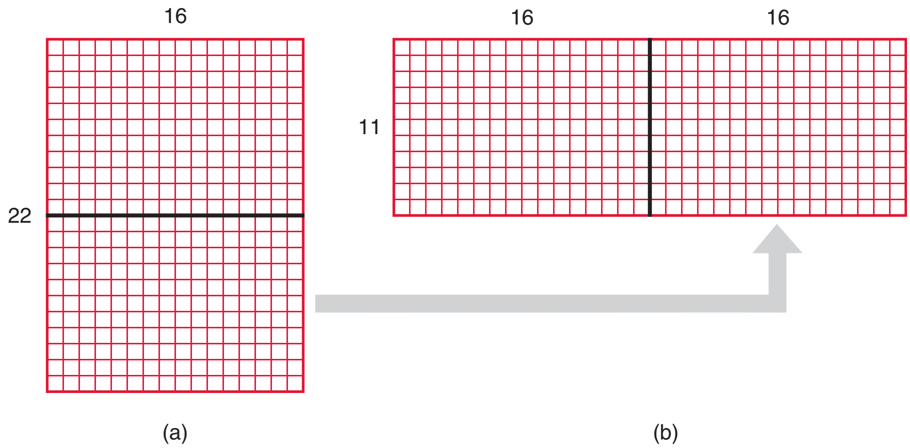


Figure 3.21

The equal-products method can also be justified by using number properties. The following equations show that $22 \times 16 = 11 \times 32$. Notice that multiplying by $\frac{1}{2}$ and 2 is the same as multiplying by 1. This is a special case of the inverse property for multiplication, which is discussed in Section 5.3.

$$
\begin{aligned}
22 \times 16 &= 22 \times 1 \times 16 && \text{identity property for multiplication} \\
&= 22 \times \left(\frac{1}{2} \times 2\right) \times 16 && \text{inverse property for multiplication} \\
&= \left(22 \times \frac{1}{2}\right) \times (2 \times 16) && \text{associative property for multiplication} \\
&= 11 \times 32
\end{aligned}
$$

EXAMPLE H

Use the method of equal products to perform the following calculations mentally.

1. 14×4
2. 28×25
3. 15×35

Solution **1.** $14 \times 4 = 7 \times 8 = 56$. **2.** $28 \times 25 = 14 \times 50 = 7 \times 100 = 700$. **3.** $15 \times 35 = 5 \times 105 = 525$.

ESTIMATION OF PRODUCTS

NCTM Standards

Instruction should emphasize the development of an estimation mindset. Children should come to know what is meant by an estimate, when it is appropriate to estimate, and how close an estimate is required in a given situation. If children are encouraged to estimate, they will accept estimation as a legitimate part of mathematics. p. 115

The importance of estimation is noted in NCTM's K–4 Standard, *Estimation* in the *Curriculum and Evaluation Standards for School Mathematics*. The techniques of *rounding,* using *compatible numbers for estimation,* and *front-end estimation* are used in the following examples.

Rounding Products can be estimated by rounding one or both numbers. Computing products by rounding is somewhat more risky than computing sums by rounding, because any error due to rounding becomes multiplied. For example, if we compute 47×28 by rounding 47 to 50 and 28 to 30, the estimated product $50 \times 30 = 1500$ is greater than the actual product. This may be acceptable if we want an estimate greater than the actual product. For a closer estimate, we can round 47 to 45 and 28 to 30. In this case the estimate is $45 \times 30 = 1350$.

EXAMPLE 1

Use rounding to estimate these products. Make any adjustments you feel might be needed.

1. 28×63
2. 81×57
3. 194×26

NCTM Standards

When students leave grade 5, . . . they should be able to solve many problems mentally, to estimate a reasonable result for a problem, . . . and to compute fluently with multidigit whole numbers. p. 149

Solution Following is one estimate for each product. You may find others. **1.** $28 \times 63 \approx 30 \times 60 = 1800$. Notice that since 63 is greater than 28, increasing 28 by 2 has more of an effect on the estimate than decreasing 63 to 60 (see Figure 3.22). So the estimate of 1800 is greater than the actual answer. **2.** $81 \times 57 \approx 80 \times 60 = 4800$. **3.** $194 \times 26 \approx 200 \times 25 = 5000$.

Figure 3.22 shows the effect of estimating 28×63 by rounding to 30×60. Rectangular arrays for both 28×63 and 30×60 can be seen on the grid. The green region shows the increase from rounding 28 to 30, and the red region shows the decrease from rounding 63 to 60. Since the green region ($2 \times 60 = 120$) is larger than the red region ($3 \times 28 = 84$), we are adding more than we are removing, and so the estimate is greater than the actual product.

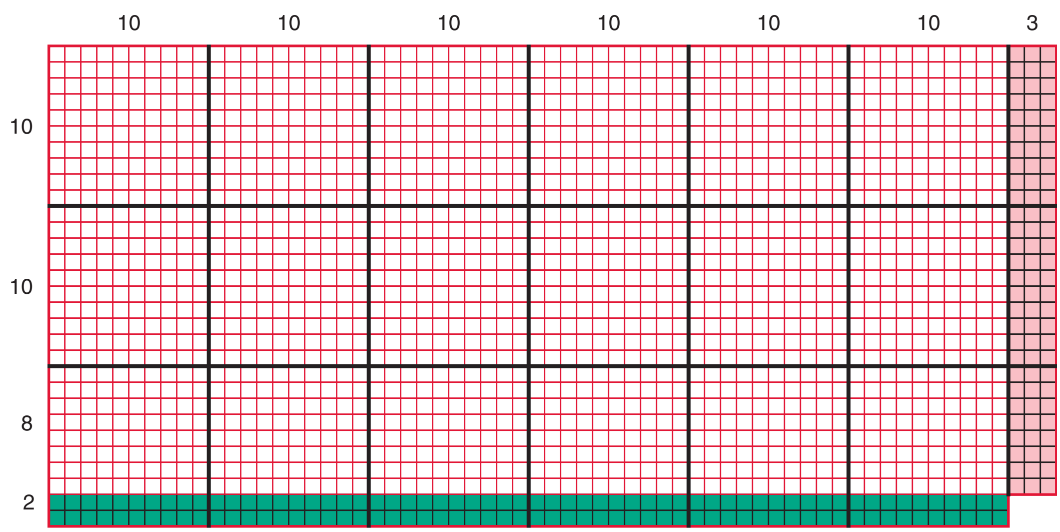


Figure 3.22

Compatible Numbers for Estimation Using compatible numbers becomes a powerful tool for estimating products when it is combined with techniques for performing mental calculations. For example, to estimate $4 \times 237 \times 26$, we might replace 26 by 25 and use a different ordering of the numbers.

$$4 \times 237 \times 26 \approx 4 \times 25 \times 237 = 100 \times 237 = 23{,}700$$

EXAMPLE J

Use compatible numbers to estimate these products.

1. $2 \times 117 \times 49$
2. $34 \times 46 \times 3$

Solution **1.** $2 \times 117 \times 49 \approx 2 \times 117 \times 50 = 100 \times 117 = 11{,}700$. **2.** $34 \times 46 \times 3 = (3 \times 34) \times 46 \approx 100 \times 46 = 4600$.

Front-End Estimation This technique is similar to that used for computing sums. The leading digit of each number is used to obtain an estimated product. To estimate 43×72, the product of the leading digits of the numbers is $4 \times 7 = 28$, so the estimated product is 2800.

$$43 \times 72 \approx 40 \times 70 = 2800$$

Similarly, front-end estimation can be used for estimating the products of numbers whose leading digits have different place values.

$$61 \times 874 \approx 60 \times 800 = 48{,}000$$

EXAMPLE K

Use front-end estimation to estimate these products.

1. 64×23 **2.** 68×87 **3.** 237×76 **4.** $30{,}328 \times 419$

Solution **1.** $64 \times 23 \approx 60 \times 20 = 1200$ **2.** $68 \times 87 \approx 60 \times 80 = 4800$ **3.** $237 \times 76 \approx 200 \times 70 = 14{,}000$ **4.** $30{,}328 \times 419 \approx 30{,}000 \times 400 = 12{,}000{,}000$

Technology Connection

Order of Operations

Special care must be taken on some calculators when multiplication is combined with addition or subtraction. The numbers and operations will not always produce the correct answer if they are entered into the calculator in the order in which they appear.

EXAMPLE L

Compute $3 + 4 \times 5$ by entering the numbers into your calculator as they appear from left to right.

Solution Some calculators will display 35, and others will display 23. The correct answer is 23 because multiplication should be performed before addition:

$$3 + 4 \times 5 = 3 + 20 = 23$$

Mathematicians have developed the convention that when multiplication occurs with addition and/or subtraction, the multiplication should be performed first. This rule is called the **order of operations.**

Technology Connection

Some calculators are programmed to follow the order of operations. On this type of calculator, any combination of products with sums and differences and without parentheses can be computed by entering the numbers and operations in the order in which they occur from left to right and then pressing [=]. If a calculator does not follow the order of operations, the products can be computed separately and recorded by hand or saved in the calculator's memory.

EXAMPLE M

Use your calculator to evaluate $34 \times 19 + 82 \times 43$. Then check the reasonableness of your answer by using estimation and mental calculations.

Solution The exact answer is 4172. An estimate can be obtained as follows:

$$34 \times 19 + 82 \times 43 \approx 30 \times 20 + 80 \times 40 = 600 + 3200 = 3800$$

Notice that the estimation in Example M is 372 less than the actual product. However, it is useful in judging the reasonableness of the number obtained from the calculator: It indicates that the calculator answer is most likely correct. If $34 \times 19 + 82 \times 43$ is entered into a calculator as it appears from left to right and if the calculator is not programmed to follow the order of operations, then the incorrect result of 31,304 will be obtained, which is too large by approximately 27,000.

PROBLEM-SOLVING APPLICATION

There is an easy method for mentally computing the products of certain two-digit numbers. A few of these products are shown here.

$$25 \times 25 = 625 \qquad 24 \times 26 = 624 \qquad 71 \times 79 = 5609$$
$$37 \times 33 = 1221 \qquad 35 \times 35 = 1225 \qquad 75 \times 75 = 5625$$

The solution to the following problem reveals the method of mental computation and uses *rectangular grids* to show why the method works.

Problem

What is the method of mental calculation for computing the products of the two-digit numbers shown above, and why does this method work?

Understanding the Problem There are patterns in the digits in these products. One pattern is that the two numbers in each pair have the same first digit. Find another pattern. **Question 1:** What types of two-digit numbers are being used?

Devising a Plan Looking for patterns may help you find the types of numbers and the method of computing. Another approach is to represent some of these products on a grid. The following grid illustrates 24×26; the product is the number of small squares in the rectangle. To determine this number, we begin by counting large groups of squares. There are 6 hundreds.

Technology Connection

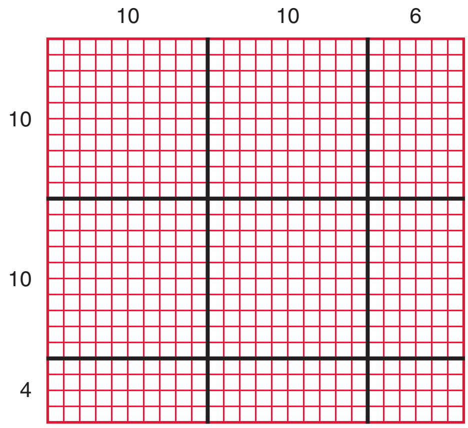
Number Chains

Start with any two-digit number, double its units digit, and add the result to its tens digit to obtain a new number. Repeat this process with each new number, as shown by the example here. What eventually happens? Use the online 3.3 Mathematics Investigation to generate number chains beginning with different numbers and to look for patterns and form conjectures.

$$14 \rightarrow 9 \rightarrow 18 \rightarrow 17 \rightarrow 15$$

Mathematics Investigation
Chapter 3, Section 3
www.mhhe.com/bbn

Question 2: Why is this grid especially convenient for counting the number of hundreds?



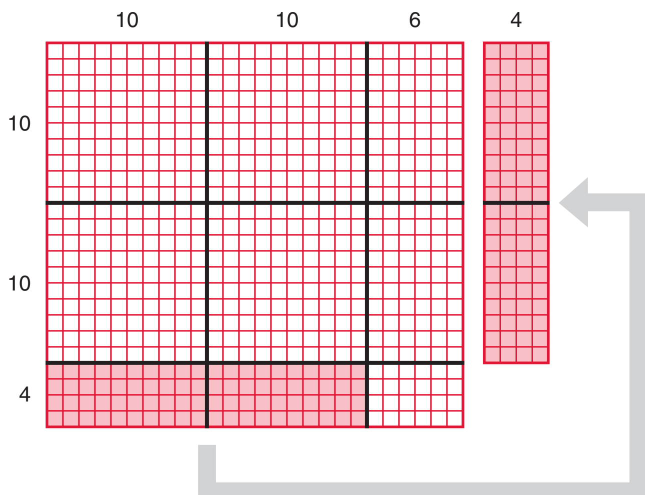
Carrying Out the Plan Sketch grids for one or more of the products being considered in this problem. For each grid it is easy to determine the number of hundreds. This is the key to solving the problem. **Question 3:** What is the solution to the original problem?

Looking Back Consider the following products of three-digit numbers:

$$103 \times 107 = 11{,}021 \qquad 124 \times 126 = 15{,}624$$

Question 4: Is there a similar method for mentally calculating the products of certain three-digit numbers?

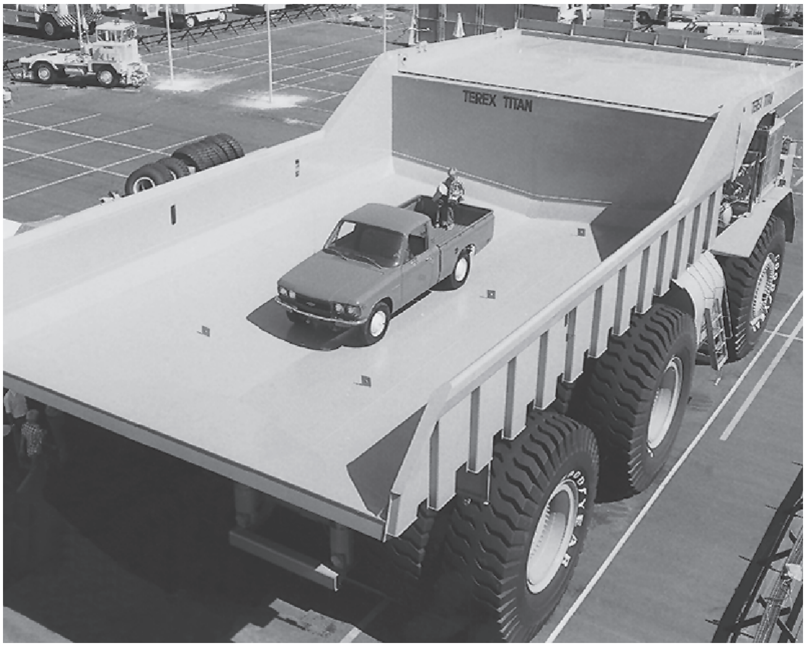
Answers to Questions 1–4 **1.** In each pair of two-digit numbers, the tens digits are equal and the sum of the units digits is 10. **2.** The two blocks of 40 squares at the bottom of the grid can be paired with two blocks of 60 squares on the right side of the grid to form two more blocks of 100, as shown below. Then the large 20×30 grid represents 6 hundreds. The 4×6 grid in the lower right corner represents 4×6.



3. The first two digits of the product are formed by multiplying the tens digit by the tens digit plus 1. The remaining digits of the product are obtained by multiplying the two units digits. **4.** Yes. For 124×126: $12 \times 13 = 156$ and $4 \times 6 = 24$, so $124 \times 126 = 15{,}624$.

Section **3.4**

DIVISION AND EXPONENTS



General Motors Terex Titan and Chevrolet Luv pickup

PROBLEM OPENER

Using exactly four 4s and only addition, subtraction, multiplication, and division, write an expression that equals each of the numbers from 1 to 10. You do not have to use all the operations, and numbers such as 44 are permitted.*

One common use of division is to compare two quantities. In the photograph above, consider the relative sizes of the Terex Titan dump truck and the Luv pickup, which is on the Titan's dump body. The Terex Titan can carry 317,250 kilograms; the Luv pickup has a limit of 450 kilograms. We can determine the number of Luv loads it requires to equal one Titan load by dividing 317,250 by 450. The answer is 705, which means the Luv pickup will have to haul 705 loads to fill the Titan just once! Sitting in the back of the Luv pickup is a child holding a toy truck. If the toy truck holds 3 kilograms of sand, how many of its loads will be required to fill the Titan?

The division operation used in comparing the sizes of the Terex Titan and the Luv pickup can be checked by multiplication. The load weight of the smaller truck times 705 should equal the load weight of the larger truck. The close relationship between division and multiplication can be used to define division in terms of multiplication.

Division of Whole Numbers For any whole numbers r and s, with $s \neq 0$, the quotient of r divided by s, written $r \div s$, is the whole number k, if it exists, such that $r = s \times k$.

*Similar equations exist for five 5s, six 6s, etc. See R. Crouse and J. Shuttleworth, "Playing with Numerals," *Arithmetic Teacher* 1, no. 5, pp. 417–419.

EXAMPLE A

Mentally calculate each quotient.

1. $18 \div 3$ **2.** $24 \div 6$ **3.** $35 \div 5$

Solution **1.** $18 \div 3 = 6$ since $18 = 3 \times 6$. **2.** $24 \div 6 = 4$ since $24 = 6 \times 4$. **3.** $35 \div 5 = 7$ since $35 = 5 \times 7$.

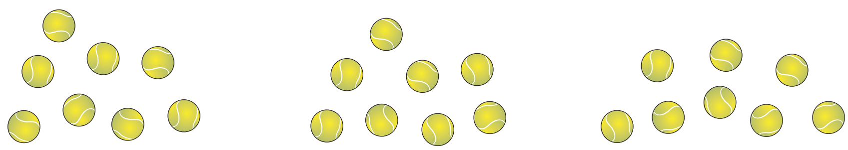
The definition of division, along with Example A, shows why multiplication and division are called **inverse operations.** We arrive at division facts by knowing multiplication facts.

There are three basic terms used in describing the division process: *dividend, divisor,* and *quotient.* In problem 1 of Example A, 18 is the **dividend,** 3 is the **divisor,** and 6 is the **quotient.** Over the centuries, division has acquired two meanings or uses. David Eugene Smith, in *History of Mathematics,* speaks of the twofold nature of division and refers to the sixteenth-century authors who first clarified the differences between its two meanings.* These two meanings of division, known as *sharing* (*partitive*) and *measurement* (*subtractive*), are illustrated in Examples B and C.

EXAMPLE B

Suppose you have 24 tennis balls, which you want to divide equally among 3 people. How many tennis balls would each person receive?

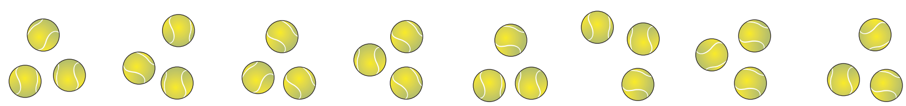
Solution The answer can be determined by separating (partitioning) the tennis balls into 3 equivalent sets. The following figure shows 24 balls divided into 3 groups and illustrates $24 \div 3 = 8$. The divisor 3 indicates the number of groups and the quotient 8 indicates the number of tennis balls in each group. This problem illustrates the **sharing (partitive) concept** of division.



EXAMPLE C

Suppose you have 24 tennis balls and want to give 3 tennis balls to as many people as possible. How many people would receive tennis balls?

Solution The answer can be determined by subtracting, or measuring off, as many sets of 3 as possible. The following figure of 24 tennis balls shows the result of this measuring process and illustrates $24 \div 3 = 8$. The divisor 3 is the number of balls in each group, and the quotient 8 is the number of groups. This problem illustrates the **measurement (subtractive) concept** of division.



NCTM Standards

In prekindergarten through grade 2, students should begin to develop an understanding of the concepts of multiplication and division. . . . They can investigate division with real objects and through story problems, usually ones involving the distribution of equal shares. p. 84

MODELS FOR DIVISION ALGORITHMS

Of the four basic pencil-and-paper algorithms, the algorithm for division, called **long division,** is the most difficult and has traditionally required the most classroom time to master. As the use of calculators in schools increases, long division, especially for three- and

* D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), p. 130.

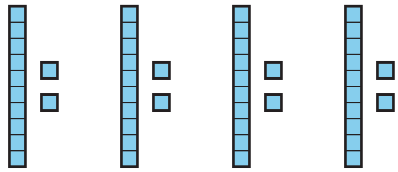
four-digit numbers, will be deemphasized. However, an understanding of division and of algorithms for determining quotients will remain important for mental calculations, estimation, and problem solving.

There are several physical models for illustrating division. Base-ten pieces are used in Examples D and E.

EXAMPLE D

Compute 48 ÷ 4 by sketching base-ten pieces.

Solution **1.** One possibility is to use the sharing (partitive) concept of division, placing 1 long in each of four groups and then 2 units in each group, as shown in the following figure. The *size* of each group, or 12, is the quotient of 48 ÷ 4.

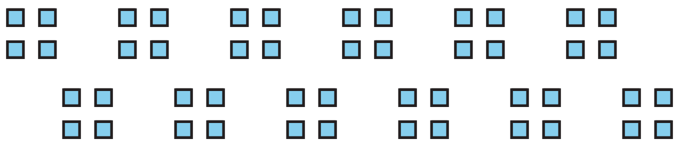


48 ÷ 4 = 12 using the sharing concept of division

NCTM Standards

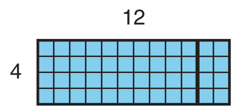
By creating and working with representations (such as diagrams or concrete objects) of multiplication and division situations, students can gain a sense of the relationships among the operations. p. 33

2. Another approach is to use the measurement (subtractive) concept of division to form as many groups of 4 units as possible. In this case there are 12 groups of 4 units each, as shown in the next figure. The *number* of groups, namely 12, is the quotient of 48 ÷ 4.



48 ÷ 4 = 12 using the measurement concept of division

3. A third possibility is to use 4 longs and 8 units to form a rectangular array with one dimension of 4, as shown next. The other dimension is 12, the quotient of 48 ÷ 4.



48 ÷ 4 = 12 using the array method of division

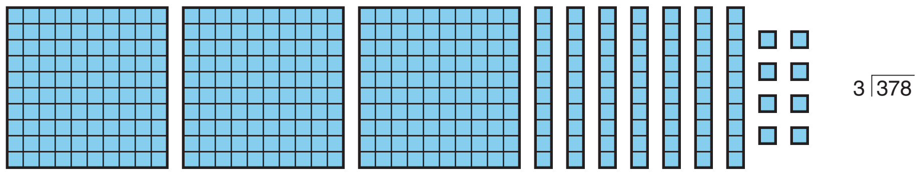
Notice in solution 3 of Example D that by viewing the rectangular array as 4 rows of 12 units each, we are making use of the sharing (partitive) concept of division for computing 48 ÷ 4, and by viewing the array as 12 columns of 4 units each, we are making use of the measurement (subtractive) concept of division for computing 48 ÷ 4.

Example E shows how base-ten pieces can be used to illustrate the steps in the long-division algorithm.

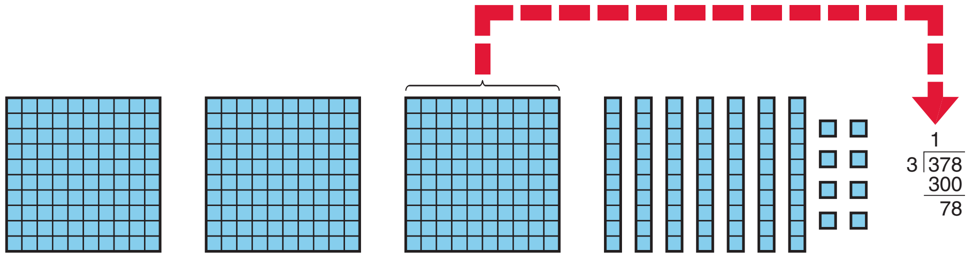
EXAMPLE E

This example illustrates 378 ÷ 3 by using the sharing (partitive) concept of division. Four steps are described. In each step, as the base-ten pieces are divided into groups, the groups are matched to the quotient of the long-division algorithm.

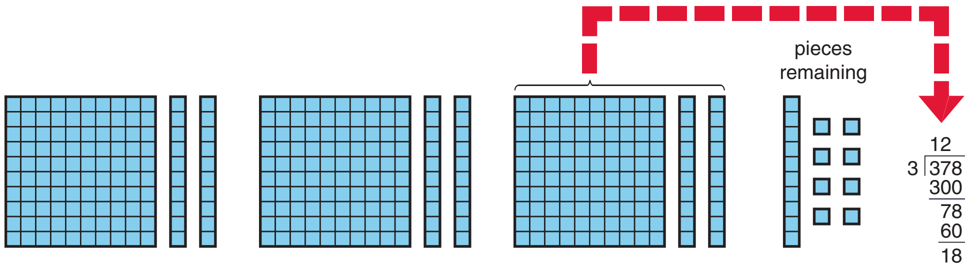
Step 1. Begin with 3 flats, 7 longs, and 8 units to represent 378.



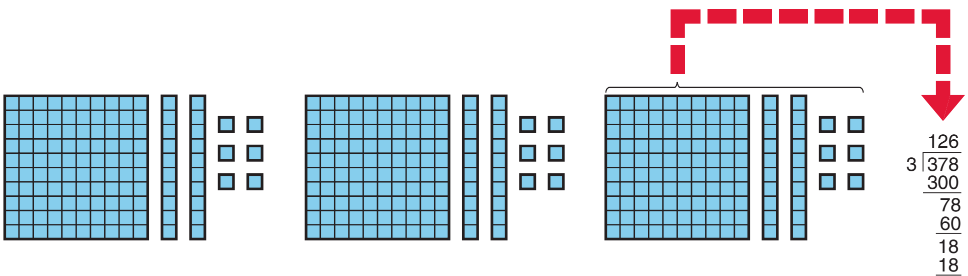
Step 2. Share the flats by placing 1 flat in each of 3 groups. This leaves 7 longs and 8 units.



Step 3. Share the longs by placing 2 longs in each of the 3 groups, leaving 1 long and 8 units.



Step 4. Regroup the remaining long into 10 units and share the 18 units by placing 6 units in each of the 3 groups.



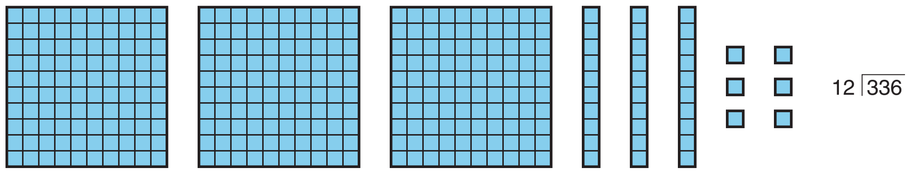
Notice that each of the final groups of base-ten pieces represents the quotient 126.

For small divisors, as in Example E, the sharing concept of division is practical because the number of groups is small. For larger divisors, as in Example F, rectangular arrays are convenient. In recent years, the rectangular-array approach to illustrating division has become more common.

EXAMPLE F

This example illustrates 336 ÷ 12 by using a rectangular array. Three steps are described, and each step is related to the quotient of the long-division algorithm.

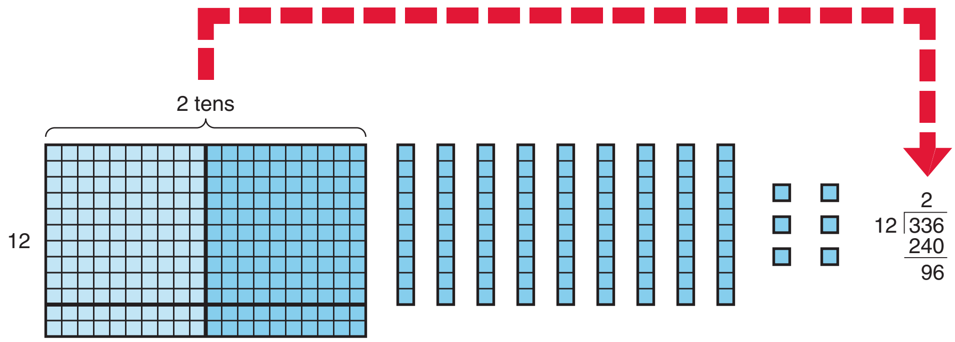
Step 1. Begin with 3 flats, 3 longs, and 6 units to represent 336.



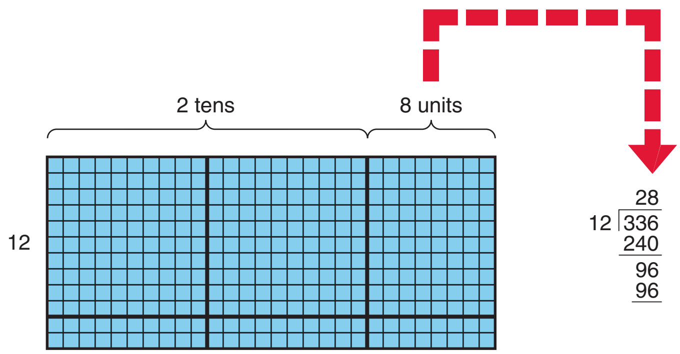
NCTM Standards

As students move from third to fifth grade, they should consolidate and practice a small number of computational algorithms for addition, subtraction, multiplication, and division that they understand well and can use routinely. p. 155

Step 2. Start building a rectangle with one dimension of 12. This can be done by beginning with 1 flat and 2 longs (see light blue region). Then a second flat and 2 more longs can be added by regrouping the third flat into 10 longs. This leaves 9 longs and 6 units.



Step 3. Continue building the rectangle by extending it with the remaining 9 longs and 6 units. To accomplish this, regroup one of the longs into 10 units.



The final dimension of the rectangle in Example F is 28, the quotient of 336 ÷ 12. Notice that the rectangular-array illustration of division is a visual reminder of the close relationship between division and multiplication: The product of the two dimensions, or 12 × 28, is 336, the number represented by the original set of base-ten pieces.

Explore

Math Activity for 8-1
Model Division

MAIN IDEA

I will explore dividing by one-digit numbers.

You Will Need
base-ten blocks

New Vocabulary
dividend
divisor
quotient
remainder

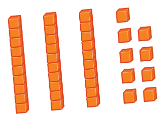
In division, the **dividend** is the number that is being divided. The **divisor** is the number that divides the dividend. The **quotient** is the result.

$$\text{divisor}\overline{\smash{)}\text{dividend}}^{\;\text{quotient}}$$

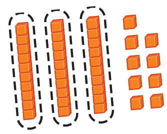
ACTIVITY

1 **Find 39 ÷ 3.**

Step 1 Model the dividend, 39.
Use 3 tens and 9 ones to show 39.

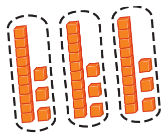


Step 2 Divide the tens.
The divisor is 3. So, divide the tens into 3 equal groups. There is a ten in each group.



$$3\overline{)39}^{\;1}$$

Step 3 Divide the ones.
Divide the ones into 3 equal groups. There are 1 ten and 3 ones in each group. So, 39 ÷ 3 = 13.



$$3\overline{)39}^{\;13}$$

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DIVISION ALGORITHM THEOREM

We have seen that the sum or product of two whole numbers is always another whole number and that this fact is called the *closure* property. Subtraction and division of whole numbers, on the other hand, are not closed. That is, the difference or quotient of two whole numbers is not always another whole number.

EXAMPLE G

1. $12 - 15$ is not a whole number because there is no whole number c such that $12 = 15 + c$.
2. $38 \div 7$ is not a whole number because there is no whole number k such that $38 = 7 \times k$.

At times we want to solve problems involving division of whole numbers even though the quotient is not a whole number. In the case of $38 \div 7$, we can determine that the greatest whole number quotient is 5 and the remainder is 3.

$$38 = 7 \times 5 + 3$$

Technology Connection

Calculators intended for schoolchildren are sometimes designed to display the whole number quotient and remainder when one whole number is divided by another. The view screens for two such calculators are shown in Figure 3.23. These screens show the quotient and remainder for $66{,}315 \div 7$. Notice that these calculators have a special key to indicate division with a whole number remainder.

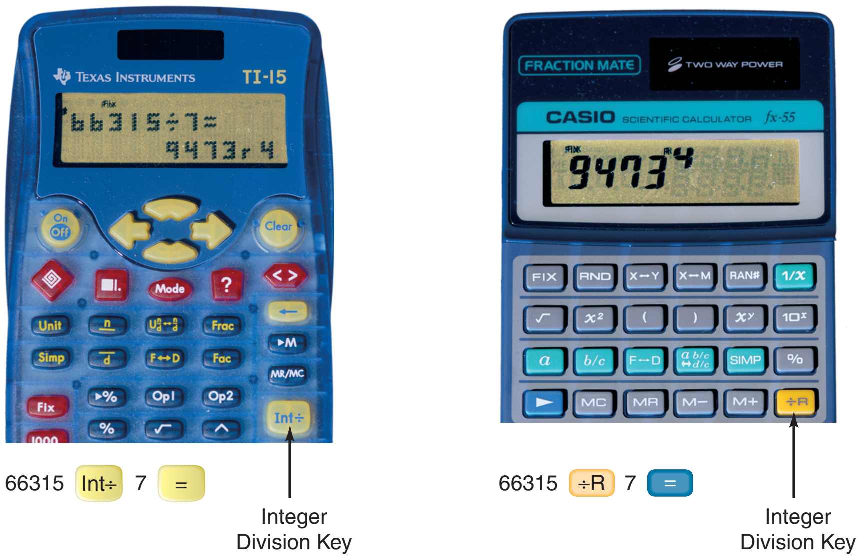

Figure 3.23

If you do not have such a calculator, whole number quotients and remainders can be obtained from any calculator. The following view screen shows the quotient in terms of a whole number and a decimal.

66315 [÷] 7 [=] [9473.571429]

The whole number quotient is 9473, and the whole number remainder can be determined as follows: $66{,}315 - 7 \times 9473 = 4$.

Technology Connection

Another approach to obtaining whole number quotients and remainders is to use repeated subtraction. This approach has the added advantage of reinforcing the measurement (subtractive) concept of division, and it can be used on most calculators. The following steps and displays show that 489 ÷ 134 has a quotient of 3 (because 134 has been subtracted 3 times) and a remainder of 87. The process of subtracting 134 continues until the view screen shows a number that is less than 134 and greater than or equal to 0.

Keystrokes	View Screen
489	489
[−] 134 [=]	355
[−] 134 [=]	221
[−] 134 [=]	87

Calculators that display whole number quotients and remainders help children to see that whenever one whole number is divided by another nonzero whole number, the quotient is always greater than or equal to 0; and the remainder is always less than the divisor.

The fact that such quotients q and remainders r always exist is guaranteed by the following theorem.

Division Algorithm Theorem For any whole numbers a and b, with **divisor** $b \neq 0$, there are whole numbers q **(quotient)** and r **(remainder)** such that

$$\boldsymbol{a = bq + r}$$

and $0 \leq \boldsymbol{r} < \boldsymbol{b}$.

This theorem says that the remainder r is always less than the divisor b. If $r = 0$, then the quotient $a \div b$ is the whole number q.

EXAMPLE H

Use a calculator and one of the preceding approaches to determine the whole number quotient and remainder.

1. 81,483 ÷ 26 **2.** 37,641 ÷ 227

3. 707,381 ÷ 112 **4.** 51,349 ÷ 57

Solution **1.** Quotient 3133 and remainder 25. **2.** Quotient 165 and remainder 186. **3.** Quotient 6315 and remainder 101. **4.** Quotient 900 and remainder 49.

MENTAL CALCULATIONS

NCTM Standards

The teacher plays an important role in helping students develop and select an appropriate computational tool (calculator, paper-and-pencil algorithm, or mental strategy). p. 156

A major strategy in performing mental calculations is replacing a problem by one that can be solved more easily. This approach was used in Sections 3.2 and 3.3 for mentally calculating sums, differences, and products; it is described here for division.

Equal Quotients for Mental Calculation In calculating a quotient mentally, sometimes it is helpful to use the method of **equal quotients,** in which we divide or multiply both the divisor and the dividend by the same number.

EXAMPLE I

1. The quotient 144 ÷ 18 can be replaced by 72 ÷ 9 by dividing both 144 and 18 by 2. We know from our basic multiplication facts that 9 × 8 = 72, so

$$144 \div 18 = 72 \div 9 = 8$$

2. The quotient 1700 ÷ 50 can be replaced by multiplying both 1700 and 50 by 2. Then dividing by 100 can be done mentally to obtain a quotient of 34.

$$1700 \div 50 = 3400 \div 100 = 34$$

The *equal quotients* calculating technique can be illustrated by the array model. Recall that this model illustrates the close connection between multiplication and division. For example, a 6 by 7 array of 42 tiles not only illustrates the product 6 × 7 = 42, but also the quotient 42 ÷ 6 = 7. Figure 3.24a shows that when both numbers in the quotient 42 ÷ 6 are divided by a number, in this case 2, the quotient remains unchanged. The first array illustrates 42 ÷ 6 and that the quotient is 7, the top side of the array. The second array is obtained by dividing the first array in half. Notice that both the left side of the array (6) and the total number of tiles (42) in the first array are cut in half to obtain the array that illustrates 21 ÷ 3. However, the top side of the second array, which is 7 (the quotient of 21 ÷ 3), has remained unchanged.

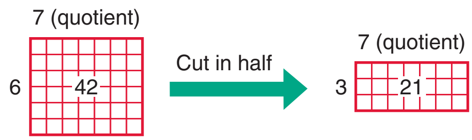

Figure 3.24a

Similarly, Figure 3.24b illustrates why both numbers in the quotient 42 ÷ 6 can be multiplied by a number, in this case 3, and the quotient remains unchanged. The first array illustrates 42 ÷ 6, and shows that the quotient is 7, as in Figure 3.24a. The second array is obtained by tripling the first array. Notice that both the left side of the array (6) and the total number of tiles (42) in the first array are tripled to obtain the array that illustrates 126 ÷ 18. Also notice that the top side of the second array, which is 7 (the quotient of 126 ÷ 18), has remained unchanged.

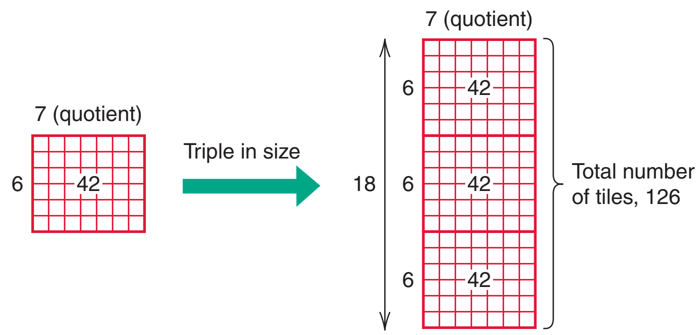

Figure 3.24b

Usually when mentally computing the quotient of two whole numbers, we want to divide both numbers by the same number to obtain smaller numbers. However, when we use the *equal-quotients* technique for mentally computing quotients of fractions and decimals, it is often more convenient to multiply both numbers in the quotient to obtain compatible numbers. (See the use of *equal quotients* in Sections 5.3 and 6.2.)

EXAMPLE J

Replace each quotient by equal quotients until you can calculate the answer mentally.

1. $180 \div 12$
2. $900 \div 36$
3. $336 \div 48$

Solution Here are three solutions. Others are possible. **1.** $180 \div 12 = 60 \div 4 = 15$ (Divide by 3). **2.** $900 \div 36 = 300 \div 12 = 100 \div 4 = 25$ (Divide by 3 twice). **3.** $336 \div 48 = 112 \div 16 = 56 \div 8 = 7$ (Divide by 3, then by 2).

ESTIMATION OF QUOTIENTS

Rounding Often we wish to obtain a rough comparison of two quantities in order to determine how many times bigger (or smaller) one is than the other. This may require finding an estimation for a quotient. *Rounding* numbers is one method of estimating a quotient.

EXAMPLE K

Mentally estimate each quotient by rounding one or both numbers.

1. $472 \div 46$
2. $145 \div 23$
3. $8145 \div 195$

Solution Here are some possibilities. **1.** $472 \div 46 \approx 460 \div 46 = 10$; or $472 \div 46 \approx 500 \div 50 = 10$. **2.** $145 \div 23 \approx 150 \div 25 = 6$. **3.** $8145 \div 195 \approx 8000 \div 200 = 40$.

HISTORICAL HIGHLIGHT

Emilie de Breteuil, 1706–1749

France, during the post-Renaissance period, offered little opportunity for the education of women. Emilie de Breteuil's precocity showed itself in many ways, but her true love was mathematics. One of her first scientific works was an investigation regarding the nature of fire, which was submitted to the French Academy of Sciences in 1738. It anticipated the results of subsequent research by arguing that both light and heat have the same cause or are both modes of motion. She also discovered that different-color rays do not give out an equal degree of heat. Her book *Institutions de physique* was originally intended as an essay on physics for her son. She produced instead a comprehensive textbook, not unlike a modern text, which traced the growth of physics, summarizing the thinking of the philosopher-scientists of her century. The work established Breteuil's competence among her contemporaries in mathematics and science.*

* L. M. Osen, *Women in Mathematics* (Cambridge, MA: The MIT Press, 1974), pp. 49–69.

Rounding to obtain an approximate quotient can be combined with the process of finding equal quotients (dividing or multiplying both the divisor and the dividend by the same number).

EXAMPLE L

Estimate each of the following by first *rounding* and then using *equal quotients.*

1. 427 ÷ 72

2. 139 ÷ 18

Solution **1.** $427 \div 72 \approx 430 \div 70 = 43 \div 7 \approx 6$. **2.** $139 \div 18 \approx 140 \div 18 = 70 \div 9 \approx 8$; or $139 \div 18 \approx 140 \div 20 = 14 \div 2 = 7$.

Compatible Numbers for Estimation Replacing numbers with *compatible numbers* is a useful technique for mentally estimating quotients.

EXAMPLE M

Find one or two compatible numbers to replace the given numbers, and mentally estimate the quotient.

1. 92 ÷ 9

2. 59 ÷ 16

3. 485 ÷ 24

Solution Here is one possibility for each quotient: **1.** $92 \div 9 \approx 90 \div 9 = 10$. **2.** $59 \div 16 \approx 60 \div 15 = 4$. **3.** $485 \div 24 \approx 500 \div 25 = 20$.

Front-End Estimation This technique can be used to obtain an estimated quotient of two numbers by using the leading digit of each number. Consider the following example, where both *compatible numbers* and *front-end estimation* are used.

$$783 \div 244 \approx 700 \div 200 = 7 \div 2 = 3\tfrac{1}{2}$$

In the example just shown, the two numbers being divided have the same number of digits. A front-end estimation also can be obtained for the quotient of two numbers when the leading digits have different place values, as in 8326 ÷ 476.

$$8326 \div 476 \approx 8000 \div 400 = 80 \div 4 = 20$$

EXAMPLE N

Use front-end estimation to estimate each quotient.

1. 828 ÷ 210

2. 7218 ÷ 2036

3. 4128 ÷ 216

Solution **1.** $828 \div 210 \approx 800 \div 200 = 8 \div 2 = 4$. **2.** $7218 \div 2036 \approx 7000 \div 2000 = 7 \div 2 = 3\tfrac{1}{2}$. **3.** $4128 \div 216 \approx 4000 \div 200 = 40 \div 2 = 20$.

Technology Connection

Estimation techniques can help check the reasonableness of results from calculator computations as noted in the *Curriculum and Evaluation Standards for School Mathematics* (p. 37).

> Estimation is especially important when children use calculators. If they need to compute 4783 ÷ 13, for example, a quick estimate can be found by using "compatible

numbers." In this case 4783 is about 4800 and 13 is about 12, so 4783 ÷ 13 is about 4800 ÷ 12. The dividing can be done mentally, since 48 and 12 are "compatible numbers" for division. Thus, 4783 ÷ 13 is about 400. This rough estimate provides children with enough information to decide whether the correct keys were pressed and whether the calculator result is reasonable.

EXPONENTS

The large numbers used today were rarely needed a few centuries ago. The word *billion,* which is now commonplace, was not adopted until the seventeenth century. Even now, billion means different things to different people. In the United States it represents 1,000,000,000 (one thousand million), and in England it is 1,000,000,000,000 (one million million).

Our numbers are named according to powers of 10. The first, second, and third powers of 10 are the familiar ten, hundred, and thousand. After this, only every third power of 10 has a new or special name: million, billion, trillion, etc.

$10^0 = 1$	one
$10^1 = 10$	ten
$10^2 = 100$	one hundred
$10^3 = 1000$	**one thousand**
$10^4 = 10{,}000$	ten thousand
$10^5 = 100{,}000$	one hundred thousand
$10^6 = 1{,}000{,}000$	**one million**
$10^7 = 10{,}000{,}000$	ten million
$10^8 = 100{,}000{,}000$	one hundred million
$10^9 = 1{,}000{,}000{,}000$	**one billion**
$10^{10} = 10{,}000{,}000{,}000$	ten billion
$10^{11} = 100{,}000{,}000{,}000$	one hundred billion
$10^{12} = 1{,}000{,}000{,}000{,}000$	**one trillion**

The operation of raising numbers to a power is called **exponentiation.**

Technology Connection

Sums and Differences

Find a pattern in the equations shown here. Does this pattern continue to hold for other two-digit numbers? Analyze this pattern and form conjectures regarding similar questions in this investigation.

$20^2 - 3^2 = 23 \times 17$

$50^2 - 7^2 = 57 \times 43$

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Exponentiation For any number ***b*** and any whole number ***n***, with ***b*** and ***n*** not both zero,

$$b^n = \underbrace{b \times b \times b \times b \times \ldots \times b}_{b \text{ occurs } n \text{ times}}$$

where ***b*** is called the **base** and ***n*** is called the **exponent.** In case ***n*** = 0 or ***n*** = 1, $b^0 = 1$ and $b^1 = b$.

EXAMPLE O

Evaluate each expression.

1. 3^4 **2.** 2^5

3. 5^0 **4.** 3^1

Solution **1.** 81 **2.** 32 **3.** 1 **4.** 3

A number written in the form b^n is said to be in **exponential form.** In general, the number b^n is called the ***n*th power** of b. The second and third powers of b, b^2 and b^3, are usually called ***b* squared** and ***b* cubed.** This terminology was inherited from the ancient Greeks, who pictured numbers as geometric arrays of dots. Figure 3.25 illustrates 2^2, a 2×2 array of dots in the form of a square, and 2^3, a $2 \times 2 \times 2$ array of dots in the form of a cube.

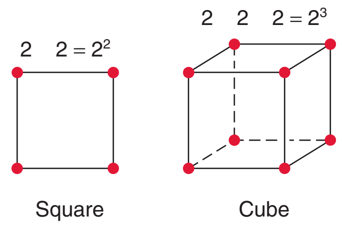


Figure 3.25

A number greater than or equal to 1 that can be written as a whole number to the second power is called a **square number** or a **perfect square** (1, 4, 9, 16, 25, . . .), and a number greater than or equal to 1 that can be written as a whole number to the third power is called a **perfect cube** (1, 8, 27, 64, 125, . . .).

Laws of Exponents Multiplication and division can be performed easily with numbers that are written as powers of the same base. To multiply, we add the exponents; to divide, we subtract the exponents.

EXAMPLE P

Evaluate each product or quotient. Write the answer in both exponential form and positional numeration.

1. $2^4 \times 2^3$ **2.** $2^8 \div 2^3$

Solution **1.** $2^4 \times 2^3 = (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^7 = 128.$

2. $2^8 \div 2^3 = \dfrac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2^5 = 32.$

The equations in Example P are special cases of the following rules for computing with exponents.

Laws of Exponents For any number a and all whole numbers m and n, except for the case where the base and exponents are both zero,

$$a^n \times a^m = a^{n+m}$$

$$a^n \div a^m = a^{n-m} \qquad \text{for } a \neq 0$$

The primary advantage of exponents is their compactness, which makes them convenient for computing with very large numbers and (as we shall see in Section 6.3) very small numbers.

EXAMPLE Q

1. In our galaxy there are 10^{11} (100 billion) stars, and in the observable universe there are 10^9 (1 billion) galaxies. If every galaxy had as many stars as ours, there would be $10^9 \times 10^{11}$ stars. Write this product in exponential form.

2. If 1 out of every 1000 stars had a planetary system, there would be $10^{20} \div 10^3$ stars with planetary systems. Write this quotient in exponential form.

3. If 1 out of every 1000 stars with a planetary system had a planet with conditions suitable for life, there would be $10^{17} \div 10^3$ such stars. Write this quotient in exponential form.

Solution **1.** 10^{20} **2.** 10^{17} **3.** 10^{14}

Technology Connection

Numbers raised to a whole-number power can be computed on a calculator by repeated multiplication, provided the products do not exceed the capacity of the calculator's view screen. On most calculators the steps shown in Figure 3.26 will produce the number represented by 4^{10}, if the process is carried out to step 10. Try this sequence of steps on your calculator.

Keystrokes	View Screen
4	4
× 4 =	16
× 4 =	64
× 4 =	256

Figure 3.26

The number of steps in the preceding process can be decreased by applying the rule for adding exponents, namely $a^n \times a^m = a^{n+m}$. To compute 4^{10}, first compute 4^5 on the calculator and then multiply the result, 1024, by itself.

$$4^{10} = 4^5 \times 4^5 = 1024 \times 1024 = 1{,}048{,}576$$

Some calculators have exponential keys such as y^x, x^y, or $\wedge$ for evaluating numbers raised to a power. To compute a number y to some exponential power x, enter the base y into the calculator, press the exponential key, and enter the exponent x. The steps in evaluating 4^{10} are shown in Figure 3.27.

Keystrokes	View Screen
4 (base)	4
y^x	4
10 (exponent)	10
=	1048576

Figure 3.27

Since 10 is a common base when exponents are used, some calculators have a key such as 10^x or 10^n. Depending on the brand of calculator, the exponent may have to be entered before pressing the exponential key or after, as shown by the following keystrokes.

5 [10^x] [100000] or [10^x] 5 [100000]

Numbers that are raised to powers frequently have more digits than the number of places in the calculator's view screen. If you try to compute 4^{15} on a calculator with only eight places in its view screen, there will not be room for the answer in positional numeration. Some calculators will automatically convert to scientific notation when numbers in positional numeration are too large for the view screen (see Section 6.3), and others will print an error message such as *Error* or *E.*

ORDER OF OPERATIONS

The rules for *order of operations,* discussed in Section 3.3, can now be extended to include division and raising numbers to powers. The order of operations requires that numbers raised to a power be evaluated first; then products and quotients are computed in the order in which they occur from left to right; finally, sums and differences are calculated in the order in which they occur from left to right. An exception to the rule occurs when numbers are written in parentheses. In this case, computations within parentheses are carried out first.

EXAMPLE R

Evaluate the following expressions.

1. $4 \times 6 + 16 \div 2^3$
2. $4 \times (6 + 16) \div 2^3$
3. $220 - 12 \times 7 + 15 \div 3$
4. $24 \div 4 \times 2 + 15$

Solution **1.** 26 (First replace 2^3 by 8; then compute the product and quotient; then add). **2.** 11 (First replace $6 + 16$ by 22; then replace 2^3 by 8; then compute the product and quotient). **3.** 141 (First replace 12×7 by 84 and $15 \div 3$ by 5; then compute the difference and sum). **4.** 27 (First compute $24 \div 4$; then multiply by 2; then add 15).

Technology Connection

Calculators that are programmed to follow the order of operations are convenient for computing expressions involving several different operations. You may wish to try problem 3 in Example R on your calculator, entering in the numbers and operations as they appear from left to right and then pressing the equality key, to see if you obtain 141.

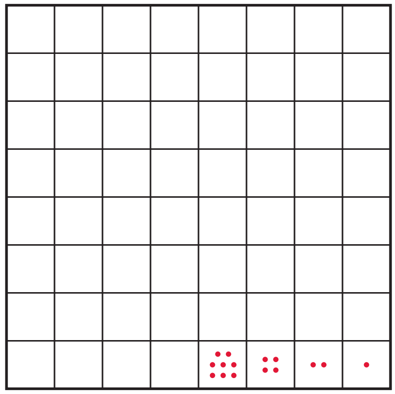
PROBLEM-SOLVING APPLICATION

The following problem involves numbers in exponential form and is solved by using the strategies of *making a table* and *finding a pattern.*

Problem

There is a legend that chess was invented for the Indian king Shirham by the grand visier Sissa Ben Dahir. As a reward, Sissa asked to be given 1 grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, then 8 grains, 16 grains, etc., until each square of the board had been accounted for. The king was surprised at such a meager request until Sissa informed him that this was more wheat than existed in the entire kingdom. What would be the sum of all the grains of wheat for the 64 squares of the chessboard?

Understanding the Problem The numbers of grains for the first few squares are shown in the following figure.



The numbers 1, 2, 4, 8, 16, 32, . . . form a geometric sequence whose common ratio is 2. Sometimes it is convenient to express these numbers as powers of 2.

$$1 \quad 2 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad \ldots$$

Question 1: How would the number of grains for the 64th square be written as a power of 2?

Devising a Plan Computing the sum of all 64 binary numbers would be a difficult task. Let's form a table for the first few sums and look for a pattern. Compute the next three totals in the following table. **Question 2:** How is each total related to a power of 2?

Square	Number of Grains	Total
1	1	1
2	$1 + 2$	3
3	$1 + 2 + 2^2$	7
4	$1 + 2 + 2^2 + 2^3$	
5	$1 + 2 + 2^2 + 2^3 + 2^4$	
6	$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$	

Carrying Out the Plan Find a pattern in the preceding table, and use it to express the sum of the grains for all 64 squares. **Question 3:** What is this sum, written as a power of 2?

Looking Back King Shirham was surprised at the total amount of grain because the number of grains for the first few squares is so small. There is more grain for each additional square than for all the preceding squares combined. **Question 4:** Why is the number of grains for the 64th square greater than the total number of grains for the first 63 squares?

Answers to Questions 1–4 **1.** 2^{63}. **2.** The total in each row is 1 less than a power of 2. **3.** The total number of grains is $2^{64} - 1$. **4.** The total number of grains for the first 63 squares is $2^{63} - 1$, but there are 2^{63} grains for the 64th square.