Section 3.2 ADDITION AND SUBTRACTION

PROBLEM OPENER

Use each of the digits 0 through 9 exactly once to obtain the smallest whole-number difference.

$$\begin{array}{r} \square\,\square\,\square\,\square\,\square \\ -\;\square\,\square\,\square\,\square\,\square \\ \hline \end{array}$$

Children learn addition at an early age by using objects. If 2 clams are *put together* with 3 clams, the total number of clams is the **sum** 2 + 3. The idea of *putting sets together,* or *taking their union,* is often used to define addition.

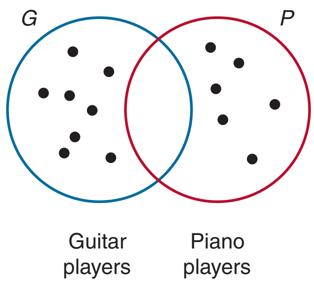
> **Addition of Whole Numbers** If set ***R*** has ***r*** elements and set ***S*** has ***s*** elements, and ***R*** and ***S*** are disjoint, then the **sum** of ***r*** plus ***s***, written ***r* + *s*,** is the number of elements in the union of ***R*** and ***S*.** The numbers ***r*** and ***s*** are called **addends.**

In the definition of addition, *R* and *S* must be disjoint sets. Otherwise, you could not determine the total number of elements in two sets by adding the number of elements in one set to the number of elements in the other set.

EXAMPLE A There are eight people in a group who play the guitar and six who play the piano. These are the only people in the group.

1. What is the minimum number of people in this group?
2. What is the maximum number of people in this group?
3. In which case (question 1 or question 2) can the answer be found by adding the number of people who play guitar to the number of people who play piano?

Solution **1.** 8 if the 6 piano players also play the guitar. **2.** 14 if the sets of piano players and guitar players are disjoint. **3.** Question 2, as illustrated on the next page.



NCTM Standards

Calculators should be available at appropriate times as computational tools, particularly when many or cumbersome computations are needed to solve problems. However, when teachers are working with students on developing computational algorithms, the calculator should be set aside to allow this focus. p. 32

MODELS FOR ADDITION ALGORITHMS

An **algorithm** is a step-by-step procedure for computing. Algorithms for addition involve two separate procedures: (1) adding digits and (2) regrouping, or "carrying" (when necessary), so that the sum is written in positional numeration. The term *carrying* probably originated back at a time when a counter, or chip, was actually carried to the next column on a counting board. Traditionally, a substantial portion of the school mathematics curriculum has involved practice with pencil-and-paper algorithms. Since calculators and computers are readily available, there can be less emphasis on written algorithms. It will always be important, however, to understand algorithms—especially in mental mathematics and estimation.

There are many models for providing an understanding of addition algorithms. Example B shows how to illustrate the sum of two numbers by using the bundle-of-sticks model. The sticks representing these numbers can be placed below each other, just as the numerals are in the addition algorithm. The sum is the total number of sticks in the bundles plus the total number of individual sticks.

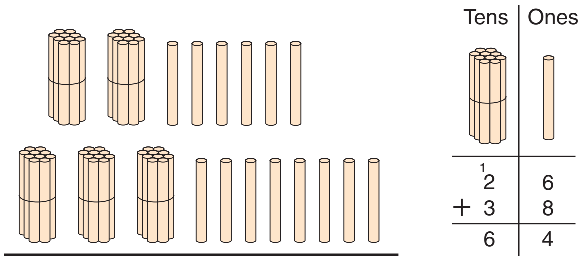
EXAMPLE B

Research Statement

Elementary school students often incorrectly employ a "when in doubt, add" strategy. This is attributed to an aspect of poorly developed conceptual knowledge.

Kroll and Miller

The numbers 26 and 38 are represented in the following figure. To compute 26 + 38, we must determine the total number of sticks. There is a total of 5 bundles of sticks (5 tens) and 14 sticks (14 ones). Since there are 14 single sticks, they can be regrouped into 1 bundle of 10 sticks and 4 more. Thus, there are a total of 6 bundles and 4 sticks. In the addition algorithm, a 4 is recorded in the units column and the extra 10 is recorded by writing a 1 in the tens column.



NCTM Standards

The use of concrete materials such as the base-ten pieces or the bundle-of-sticks model provides opportunities for students to develop their own methods of computing. The *Curriculum and Evaluation Standards for School Mathematics* (p. 95) recognizes the value of such activities:

> As they begin to understand the meaning of operations and develop a concrete basis for validating symbolic processes and situations, students should design their own algorithms and discuss, compare, and evaluate them with their peers and teacher.

Students using the model in Example B might find it natural to combine all the single sticks first, next combine the bundles of 10, and then do the regrouping. This can lead to an algorithm called **partial sums.** In this method, the digits for each place value are added, and the partial sums are recorded before there is any regrouping.

Two methods of writing partial sums are shown in Example C. In part 1 there is seldom a need for regrouping, because if there is more than one digit in the partial sum, the digits are placed in different columns. In part 2 the regrouping can be done beginning with any partial sum with more than one digit.

EXAMPLE C

1.

$$\begin{array}{r} 345 \\ +\ 278 \\ \hline 13 \\ 11 \\ 5 \\ \hline 623 \end{array}$$

2.

$$\begin{array}{lrcl} & 345 & = & 3 \text{ hundreds} + 4 \text{ tens} + 5 \\ & +278 & = & 2 \text{ hundreds} + 7 \text{ tens} + 8 \\ \cline{4-4} & & & 5 \text{ hundreds} + 11 \text{ tens} + 13 \\ \text{Regrouping:} & & & 6 \text{ hundreds} + 2 \text{ tens} + 3 \\ & & = & 623 \end{array}$$

Left-to-Right Addition Some students might begin the process of combining the sticks in Example B by first combining the bundles of 10. Since children learn to read from left to right, some may find it natural to add in this direction. The next example illustrates this process in computing the sum of two three-digit numbers.

EXAMPLE D

To compute 897 + 537 from left to right, we first add 8 and 5 in the hundreds column (see below). In the second step, 9 and 3 are added in the tens column, and because regrouping (carrying) is necessary, 3 in the hundreds column is scratched out and replaced by 4. In the third step, we add the units digits. Again regrouping is necessary, so 2 in the tens column is scratched out and replaced by 3.

$$\begin{array}{c} \text{First step} \\ \begin{array}{r} 897 \\ +537 \\ \hline 13 \\ \end{array} \end{array} \qquad \begin{array}{c} \text{Second step} \\ \begin{array}{r} 897 \\ +537 \\ \hline 1\cancel{3}2 \\ 4 \end{array} \end{array} \qquad \begin{array}{c} \text{Third step} \\ \begin{array}{r} 897 \\ +537 \\ \hline 1\cancel{3}\cancel{2}4 \\ 4\,3 \end{array} \end{array}$$

The early Hindus and later the Europeans added from left to right. The Europeans called this algorithm the **scratch method.**

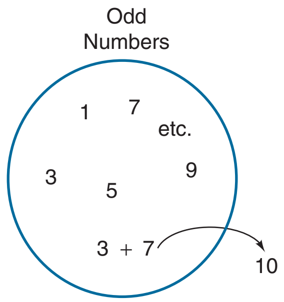
NUMBER PROPERTIES

A few fundamental properties for operations on whole numbers are so important that they are given special names. Four properties for addition are introduced here, and the corresponding properties for multiplication are given in Section 3.3.

NCTM Standards

Research has shown that learning about number and operations is a complex process for children (e.g., Fuson). p. 32

Closure Property for Addition If you were to select any two whole numbers, their sum would be another whole number. This fact is expressed by saying that the whole numbers are **closed for the operation of addition.** In general, the word *closed* indicates that when an operation is performed on any two numbers from a given set, the result is also in the set, rather than outside the set. For example, the set of whole numbers is not closed for subtraction, because sometimes the difference between two whole numbers is a negative number. Consider another example. If we select any two numbers from the set of odd numbers {1, 3, 5, 7, . . . }, the sum is not another odd number. So the set of odd numbers is not closed for addition. To test for closure, students sometimes find it helpful to draw a circle and write the numbers from a given set inside. Then if the set is *closed*, the results of the operation will be inside the circle. If the given operation produces *at least one* result that is outside the circle, the set is *not closed* for the given operation.



Closure Property For every pair of numbers in a given set, if an operation is performed, and the result is also a number in the set, the set is said to be **closed for the operation.** If one example can be found where the operation does not produce an element of the given set, then the set is **not closed for the operation.**

EXAMPLE E

Determine whether the set is closed or not closed for the given operation.

1. The set of odd numbers for subtraction.
2. The set of odd numbers for multiplication.
3. The set of whole numbers for division.

Solution **1.** The set of odd numbers is not closed for subtraction. For example, $23 - 3$ is not an odd number. **2.** The set of odd numbers is closed for multiplication; the product of any two odd numbers is another odd number. **3.** The set of whole numbers is not closed for division. For example, $\frac{2}{3}$ is not a whole number.

Identity Property for Addition Included among the whole numbers is a very special number, zero. Zero is called the **identity for addition** because when it is added to another number, there is *no change.* That is, adding 0 to any number leaves the identity of the number unchanged. For example,

$$0 + 5 = 5 \qquad 17 + 0 = 17 \qquad 0 + 0 = 0$$

Zero is unique in that it is the only number that is an identity for addition.

Identity Property for Addition For any whole number ***b*,**

$$\mathbf{0 + \boldsymbol{b} = \boldsymbol{b} + 0 = \boldsymbol{b}}$$

and 0 is the unique identity for addition.

Associative Property for Addition In any sum of three numbers, the middle number may be added to (associated with) either of the two end numbers. This property is called the **associative property for addition.**

EXAMPLE F

$$\underbrace{147 + (20 + 6)} = \underbrace{(147 + 20) + 6}$$

Associative property for addition

> **Associative Property for Addition** For any whole numbers ***a, b,*** and ***c,***
>
> $$a + (b + c) = (a + b) + c$$

When elementary school students compute by breaking a number into a convenient sum, as in Example G, the *associative property of addition* plays a role. Arranging numbers to produce sums of 10 is called *making 10s.*

EXAMPLE G

$$8 + 7 = \underbrace{8 + (2 + 5)} = \underbrace{(8 + 2) + 5} = 10 + 5 = 15$$

Associative property for addition

Commutative Property for Addition When two numbers are added, the numbers may be interchanged (commuted) without affecting the sum. This property is called the **commutative property for addition.**

EXAMPLE H

$$\underbrace{257 + 498} = \underbrace{498 + 257}$$

Commutative property
for addition

> **Commutative Property for Addition** For any whole numbers ***a*** and ***b,***
>
> $$a + b = b + a$$

As the addition table in Figure 3.5 on the next page shows, the commutative property for addition roughly cuts in half the number of basic addition facts that elementary school students must memorize. Each sum in the shaded part of the table has a corresponding equal sum in the unshaded part of the table.

Technology Connection

Palindromic Differences

Begin with any three-digit number, reverse its digits, and subtract the smaller from the larger. If this process is continued, will the result eventually be a palindromic number? The online 3.2 Mathematics Investigation will carry out this reversing and subtracting process and quickly supply you with data for making conjectures.

Mathematics Investigation
Chapter 3, Section 2
www.mhhe.com/bbn

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 3.5

EXAMPLE I

If we know that $3 + 8 = 11$, then, by the commutative property for addition, $8 + 3 = 11$. What do you notice about the locations of these sums in the addition table?

Solution The sums of $3 + 8$ and $8 + 3$ are in opposite parts of the table. If the shaded part of the table is folded onto the unshaded part of the table, these sums will coincide. That is, the table is symmetric about the diagonal from upper left to lower right.

The commutative property also enables us to select convenient combinations of numbers when we are adding.

EXAMPLE J

The numbers 26, 37, and 4 are arranged more conveniently on the right side of the following equation than on the left, because $26 + 4 = 30$ and it is easy to compute $30 + 37$.

$$26 + \underbrace{37 + 4} = 26 + \underbrace{4 + 37} = (26 + 4) + 37 = 30 + 37$$

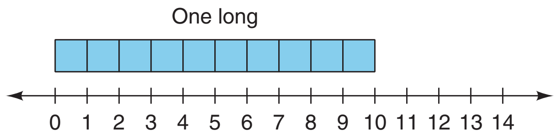
Commutative property
for addition

INEQUALITY OF WHOLE NUMBERS

The inequality of whole numbers can be understood intuitively in terms of the locations of numbers as they occur in the counting process. For example, 3 is less than 5 because it is named before 5 in the counting sequence. This ordering of numbers can be illustrated with a number line. A **number line** is formed by beginning with any line and marking off two points, one labeled 0 and the other labeled 1, as shown in Figure 3.6 on the next page. This **unit segment** is then used to mark off equally spaced points for consecutive whole numbers. For any two numbers, the one that occurs on the left is less than the one that occurs on the right.

One method of marking off unit lengths to form a number line is to use the edges of base-ten pieces, such as the *long,* for marking off 10 units (see Figure 3.6). This use of base-ten pieces provides a link between the region model and the linear model for illustrating numbers.

Figure 3.6



The inequality of whole numbers is defined in terms of addition.

Inequality of Whole Numbers For any two whole numbers m and n, m is less than n (written $m < n$) if and only if there is a nonzero whole number k such that $m + k = n$.

An inequality can be written with the inequality symbol opening to the right or to the left. For example, $4 < 9$ means that 4 is **less than** 9; and $9 > 4$ means that 9 is **greater than** 4. Sometimes the inequality symbol is combined with the equality symbol: $\leq$ means **less than or equal to,** and $\geq$ means **greater than or equal to.**

HISTORICAL HIGHLIGHT

The symbols $<$ and $>$ were first used by English surveyor Thomas Harriot in 1631. There is no record of why Harriot chose these symbols, but the following conjecture is logical and will help you to remember their meanings. The distances between the ends of the bars in the equality symbol are equal, and in an equation (for example, $3 = 1 + 2$) the number on the left of the equality symbol equals the number on the right. Similarly, $3 < 4$ indicates that 3 is less than 4, because the distance between the bars on the left is less than the distance between the bars on the right. The reasoning is the same whether we write $3 < 4$ or $4 > 3$. These symbols could easily have evolved into our present notation, $<$ and $>$, in which the bars completely converge to prevent any misjudgment of the distances.*

*This is one of two conjectures on the origin of the inequality symbols, described by H. W. Eves in *Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 111–113.

Research Statement

For students in grades K–2, learning to see the part to whole relations in addition and subtraction situations is one of their most important accomplishments in arithmetic.

Resnick

MODELS FOR SUBTRACTION ALGORITHMS

Subtraction is often explained as the *taking away* of a subset of objects from a given set. The word *subtract* literally means to *draw away from under.*

The process of taking away, or subtraction, may be thought of as the opposite of the process of putting together, or addition. Because of this dual relationship, subtraction and addition are called **inverse operations.** This relationship is used to define subtraction in terms of addition.

Subtraction of Whole Numbers For any whole numbers r and s, with $r \geq s$, the **difference** of r **minus** s, written $r - s$, is the whole number c such that $r = s + c$. The number c is called the **missing addend.**

NCTM Standards

By the end of grade 2, children should know the basic addition and subtraction combinations, should be fluent in adding two-digit numbers, and should have methods for subtracting two-digit numbers. p. 33

The definition of subtraction says that we can compute the difference $17 - 5$ by determining the **missing addend,** that is, finding the number that must be added to 5 to give 17. Store clerks use this approach when making change. Rather than subtracting 83 cents from \$1.00 to determine the difference, they pay back the change by counting up from 83 to 100.

After negative numbers are introduced, there is no need to require r to be greater than or equal to s in the definition of subtraction. In the early school grades, however, before negative numbers appear, most examples involve subtracting a smaller number from a larger one.

Three concepts of subtraction occur in problems: the **take-away concept,** the **comparison concept,** and the **missing addend concept.**

Take-Away Concept Suppose that you have 12 stamps and give away 7. How many stamps will you have left? Figure 3.7 illustrates $12 - 7$ by showing 7 objects being taken away from 12 objects.

Figure 3.7 Take-away concept showing $12 - 7 = 5$

Comparison Concept Suppose that you have 12 stamps and someone else has 7 stamps. How many more stamps do you have than the other person? In this case we compare one collection to another to determine the difference. Figure 3.8 shows that there are 5 more stamps in one collection than in the other

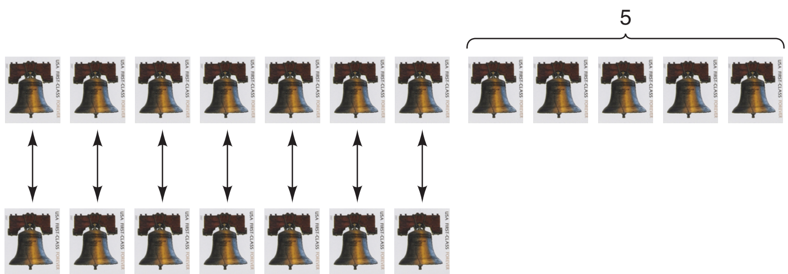


Figure 3.8 Comparison concept showing $12 - 7 = 5$

Missing Addend Concept Suppose that you have 7 stamps and you need to mail 12 letters. How many more stamps are needed? In this case we can count up from 7 to 12 to determine the missing addend. Figure 3.9 shows that 5 stamps should be added to 7 stamps to form a collection of 12 stamps.

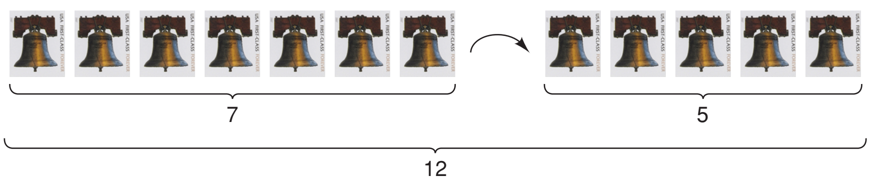


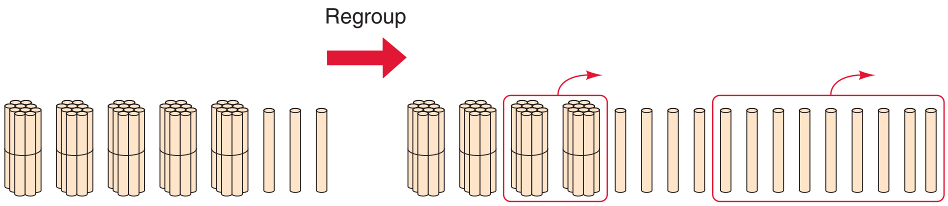
Figure 3.9 Missing addend concept showing $12 - 7 = 5$

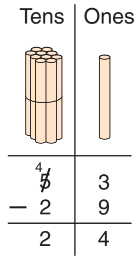
There are two types of examples to consider in explaining the steps in finding the difference between two multidigit numbers: examples in which regrouping (borrowing) is not needed and those in which regrouping (borrowing) is needed.

The bundle-of-sticks model and the take-away concept of subtraction are used in Example K to illustrate the subtraction algorithm with regrouping.

EXAMPLE K

To illustrate 53 − 29, we begin with 5 bundles of sticks (5 tens) and 3 sticks (3 ones), as shown. To take away 9 sticks, we must regroup one bundle, to form 13 single sticks. Once this has been done, we can take away 2 bundles of sticks and 9 sticks, leaving 2 bundles of sticks and 4 single sticks. In the algorithm, the regrouping is recorded by crossing out 5 and writing 4 above it.





Technology Connection

Sums and differences can be computed on calculators with algebraic logic by entering the numbers from left to right as they occur in equation form. For instance, 475 + 381 − 209 is computed by the following key strokes.

Keystrokes	View Screen
475	475
+	475
381	381
−	856
209	209
=	647

When numbers and operations are entered into some calculators, such as the one in Figure 3.10, they are displayed on the view screen from left to right as illustrated. If more numbers and operations are entered than can be displayed on the view screen of this calculator, previous entries are pushed off the left end of the screen but are retained internally in the calculator's memory.

Figure 3.10

14-8

1 Introduce

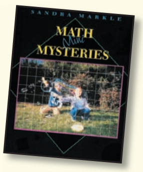
Activity Choice 1 • Hands-On

Write the number 642 on the board. Give each student eleven 10 × 10 squares of grid paper.

- **How many small squares are there on each large square?** 100
- Have students cut single squares to represent ones and columns of squares to represent tens.
- Have them model the number 642.
- Ask students to suggest other numbers to make with their squares.

Activity Choice 2 • Literature

Introduce the lesson with "You and the Statue" by Sandra Markle from *Math Mini Mysteries*. For a related math activity, see p. TR65.



2 Teach

Use place-value workmats and base-ten blocks.

- **Can you subtract 2 from 4?** yes
- Have students put 4 units in the ones column on their place-value chart. **Can you subtract 5 from 3?** no **What will you have to do?** regroup hundreds
- Have students trade a flat for 10 rods. **What number have you made?** 434
- Have students move one yellow counter into the tens column. **What number is in the tens column?** 13 (130)
- **Can you subtract 5 rods from 13 rods? (50 from 130)** yes Point out that you have regrouped hundreds to solve the problem.

Get Ready Use the section at the top of p. 457 to reinforce the lesson concept. Guide students in regrouping hundreds.

Check Observe students as you work through Exercises 1–4 as a class.

Exercise 5 Assess student comprehension before assigning practice exercises.

COMMON ERROR!

If students have trouble adding the regrouped hundred to the number that is already in the tens place, have them write out the addition problem in the margin. For example: 4 tens plus 10 tens, or 40 plus 100, equals 140.

Name ______________________

Hands-On Activity

Regroup Hundreds

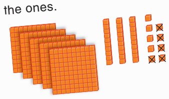
Get Ready

Main Idea

I will regroup hundreds to subtract three-digit numbers.

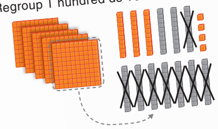
Find 539 – 285.

Step 1: Subtract the ones.



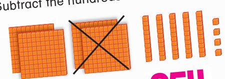
	hundreds	tens	ones
	5	3	9
–	2	8	5
			4

Step 2: Subtract the tens. You cannot subtract 8 from 3. Regroup 1 hundred as 10 tens.



	hundreds	tens	ones
	4	13	
	~~5~~	~~3~~	9
–	2	8	5
		5	4

Step 3: Subtract the hundreds.



	hundreds	tens	ones
	4	13	
	~~5~~	~~3~~	9
–	2	8	5
	2	5	4

539 – 285 = 254

Check

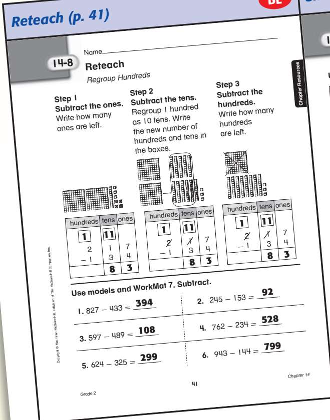
5. Sample answer: Regroup when there are not enough ones or tens to subtract.

Use base-ten blocks and WorkMat 7. Subtract.

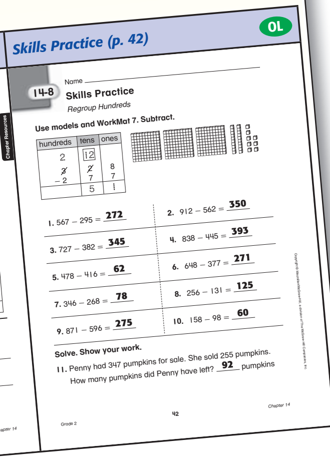
1. 568 – 197 = 371
2. 628 – 442 = 186
3. 759 – 5 = 754
4. 341 – 22 = 319
5. Talk About It How do you know when to regroup?

Chapter 14 Lesson 8 four hundred fifty-seven 457

Reteach (p. 41) BL



Skills Practice (p. 42) OL



Calculators can be used to strengthen students' understanding of place value and algorithms for computing. Earlier in this section we discussed partial sums and left-to-right addition. The next keystrokes illustrate these methods for computing $792 + 485 + 876$. Notice that the first view screen shows the sum of the hundreds; the second screen shows the sum of the hundreds and tens; and the last screen shows the sum of the original three numbers.

Keystrokes	View Screen
700 + 400 + 800 +	1900
90 + 80 + 70 +	2140
2 + 5 + 6 =	2153

MENTAL CALCULATIONS

Mental calculations are important because they often prove the quickest and most convenient method of obtaining an answer. Performing mental computations requires us to combine a variety of skills: the abilities to use various algorithms, to understand place value and base-ten numeration, and to use number properties. Mental calculations are useful in obtaining exact answers, and they are a prerequisite to estimating. Let's consider a few techniques for performing mental calculations.

Compatible Numbers for Mental Calculation One mental calculating technique is to look for pairs of numbers whose sum or difference is easy to compute. For example, it is convenient to combine 17 and 43 in the following computation.

$$17 - 12 + 43 = 17 + 43 - 12 = 60 - 12 = 48$$

Using pairs of numbers that are especially easy to compute with is the calculating technique called **compatible numbers for mental calculations.**

EXAMPLE L

Do the following computations using compatible numbers for mental calculations.

1. $17 + 12 + 23 + 45$

2. $12 - 15 + 82 - 61 + 55$

Solution **1.** One possibility is to notice that $17 + 23 = 40$; then $40 + 45 = 85$, and adding 12 produces 97. Another possibility is to notice that $12 + 23 = 35$. Then $35 + 45 = 80$, and adding 17 produces 97. **2.** Here is one possibility: $55 - 15 = 40$ and $82 - 61 = 21$. Then $40 + 21 = 61$, and adding 12 produces 73.

Substitutions for Mental Calculation Using the method of **substitutions,** a number is broken down into a convenient sum or difference of numbers. You can easily compute the sum $127 + 38$ in your head in many ways. Here are three possibilities:

$$\begin{aligned}
127 + (3 + 35) &= (127 + 3) + 35 = 130 + 35 = 165\\
127 + (30 + 8) &= (127 + 30) + 8 = 157 + 8 = 165\\
(125 + 2) + 38 &= 125 + (2 + 38) = 125 + 40 = 165
\end{aligned}$$

EXAMPLE M

Do each computation mentally by substituting a convenient sum or difference for one of the given numbers.

1. 57 + 24

2. 163 − 46

Solution Here is one possibility for each computation: **1.** 57 + 20 + 4 = 77 + 4 = 81. **2.** 163 − 40 − 6 = 123 − 6 = 117.

Equal Differences for Mental Calculation The method of **equal differences** uses the fact that the difference between two numbers is unchanged when both numbers are increased or decreased by the same amount. Figure 3.11 illustrates why this is true when both numbers are increased. No matter how many tiles (see blue tiles) are adjoined to the two rows in this figure, the difference between the numbers of tiles in the two rows is 11 − 7 = 4.

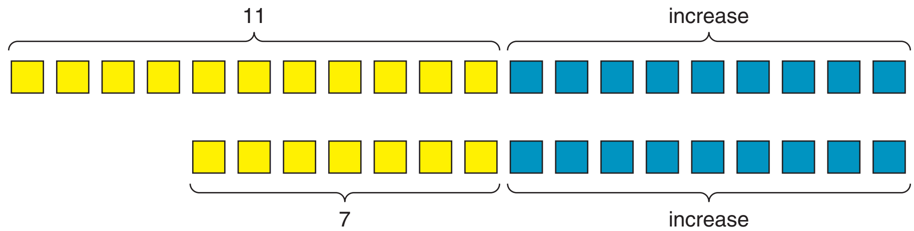


Figure 3.11

Replacing a difference by an equal but more convenient difference can be very useful.

EXAMPLE N

To compute 47 − 18, first find a more convenient but equal difference by increasing or decreasing both numbers by the same amount.

Solution Here are several differences that are more convenient for computing 47 − 18.

49 − 20	(both numbers were increased by 2)
50 − 21	(both numbers were increased by 3)
30 − 1	(both numbers were decreased by 17)
40 − 11	(both numbers were decreased by 7)

The difference, 29, is easy to compute in any of these forms.

Add-Up Method for Mental Calculation A convenient mental method for subtracting is to **add up** from the smaller to the larger number, using several easy steps. For example, to compute 54 − 19, first add 1 to 19 to obtain 20 and then 34 to 20 to obtain 54. The difference is the sum of the "add ups": 54 − 19 = 1 + 34 = 35.

EXAMPLE O

Compute each difference by adding up from the smaller to the larger number.

1. 53 − 17

2. 135 − 86

Solution **1.** From 17 to 20 is 3, and from 20 to 53 is 33. So the difference is 3 + 33 = 36. **2.** From 86 to 100 is 14, and from 100 to 135 is 35. So the difference is 14 + 35 = 49.

ESTIMATION OF SUMS AND DIFFERENCES

In recent years, the teaching of estimation has become a top priority in school mathematics programs. Often in everyday applications we need to make a quick calculation that does not have to be exact to serve the purpose at hand. For example, when shopping, we may want to estimate the total cost of the items selected in order to avoid an unpleasant surprise at the checkout counter. Estimation is especially important for developing "number sense" and predicting the reasonableness of answers. With the increased use of calculators, estimation helps students to determine if the correct keys have been pressed.

There are some difficulties in teaching estimation. First, the best estimating technique to use often depends on the numbers involved and the context of the problem. Second, there is no correct answer. An estimate is a "ballpark" figure, and for a given problem there will often be several different estimates.

There are many techniques for estimating. Three common ones—*rounding, using compatible numbers for estimation,* and *front-end estimation*—are explained below. After obtaining an estimation, we sometimes need to know if it is less than or greater than the actual answer. This can often be determined from the method of estimation used.

Rounding If an approximate sum or difference is all that is needed, we can round the numbers before computing. The type of problem will often determine to what place value the numbers will be rounded. The following estimates are obtained by rounding to the nearest hundreds or thousands. The symbol $\approx$ means **approximately equal to.**

EXAMPLE P

Obtain an estimation by rounding each number to the place value of the leading digit.

1. $624 - 289 - 132$
2. $4723 + 419 + 1040$
3. $812 - 245$

Solution **1.** $\approx 600 - 300 - 100 = 200$ **2.** $\approx 5000 + 400 + 1000 = 6400$ **3.** $\approx 800 - 200 = 600$

Some people prefer rounding each number to the same place value. If each number in part 2 of Example P were rounded to the nearest thousand, 419 would be rounded to 0 and the approximate sum would become $5000 + 0 + 1000 = 6000$. Even when numbers have the same number of digits, they do not have to be rounded to the same place value. A different estimation could be obtained in part 3 of Example P by rounding 245 to 250 (the nearest ten). We could then use the add-up method to obtain a difference of 550.

$$812 - 245 \approx 800 - 250 = 550$$

Compatible Numbers for Estimation Sometimes a computation can be simplified by replacing one or more numbers by approximations in order to obtain **compatible numbers.** For example, to approximate $342 + 250$, we might replace 342 by 350.

$$342 + 250 \approx 350 + 250 = 600$$

Using compatible numbers is a common estimating technique.

EXAMPLE Q

Use compatible numbers for estimation to obtain each sum or difference. Without computing the actual answer, predict whether your estimate is too small or too big.

1. $88 + 37 + 66 + 24$
2. $142 - 119$
3. $127 + 416 - 288$

Solution Here are some estimations. Others may occur to you. **1.** $90 + 40 + 70 + 20 = 220$, which is greater than the actual answer. **2.** $140 - 120 = 20$, which is less than the actual answer. **3.** $130 + 400 - 300 = 230$, which is less than the actual answer.

Front-End Estimation The method of **front-end estimation** is similar to left-to-right addition, but involves only the leading digit of each number.

Suppose you have written checks for \$433, \$684, and \$228 and wish to quickly estimate the total. Using front-end estimation, we see that the sum of the leading digits is 12, so the estimated sum is 1200.

$$433 + 684 + 228 \approx 400 + 600 + 200 = 1200$$

This method of estimation is different from rounding to the highest place value. For example, in the preceding sum, 684 is replaced by 600, rather than the rounded value of 700.

The next example shows how front-end estimation is used when the leading digit of each number in a sum does not have the same place value.

$$3827 + 458 + 5031 + 311 \approx 3000 + 400 + 5000 + 300 = 8700$$

Notice that in estimating the sums in these two examples, the digits beyond the leading digit of each number are not used. Thus, when front-end estimation is used for sums, the estimation is always less than or equal to the exact sum.

Front-end estimation is used for estimating both sums and differences in Example R.

EXAMPLE R

Use front-end estimation to estimate each sum or difference.

1. $1306 + 7247 + 3418$

2. $4718 - 1335$

3. $527 + 4215 + 718$

4. $7316 - 547$

Solution **1.** $1306 + 7247 + 3418 \approx 1000 + 7000 + 3000 = 11{,}000$ **2.** $4718 - 1335 \approx 4000 - 1000 = 3000$ **3.** $527 + 4215 + 718 \approx 500 + 4000 + 700 = 5200$ **4.** $7316 - 547 \approx 7000 - 500 = 6500$

Large errors from computing on a calculator, such as those produced by pressing an incorrect key, can sometimes be discovered by techniques for estimating. Suppose, for example, that you wanted to add 417, 683, and 228, but you entered 2228 on the calculator rather than 228. The sum of the three numbers you intended to add when rounded to the nearest hundred is 1300, but the erroneous calculator sum will be 3328. The difference of more than 2000 between the estimation and the calculator sum indicates that the computation should be redone.

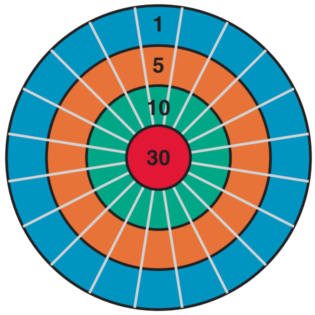
$$\begin{array}{rcr}
\text{Sum} & & \text{Estimation} \\
 & \text{(rounding)} & \\
417 & \longrightarrow & 400 \\
683 & \longrightarrow & 700 \\
\underline{+\ 228} & \longrightarrow & \underline{+\ 200} \\
 & & 1300
\end{array}$$

PROBLEM-SOLVING APPLICATION

The following problem introduces the strategy of **making an organized list.** This problem-solving strategy is closely associated with another strategy called *eliminating possibilities.* Next to guessing and checking, one of the most common approaches to solving problems is to systematically search for or eliminate possibilities.

Problem

Karen and Angela are playing darts on the board shown below. Each player throws three darts on her turn and adds the numbers on the regions that are hit. The darts always hit the dartboard, and when a dart lands on a line, the score is the larger of the two numbers. After four turns Karen and Angela notice that their sums for each turn are all different. How many different sums are possible?



Understanding the Problem **Question 1:** What are the largest and smallest possible sums?

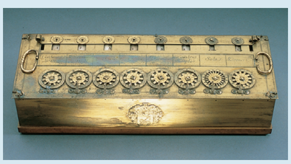
Devising a Plan Here are two approaches to finding all the sums. Since the lowest sum is 3 and the highest sum is 90, we can list the numbers from 3 through 90 and determine which can be obtained. Or we can *make an organized list* showing the different regions the three darts can strike. **Question 2:** For example, if the first two darts land in regions 1 and 5, what are the possible scores after the third dart is thrown?

Carrying Out the Plan Use one of the above approaches or one of your own to find the different sums and determine how each can be obtained from the dartboard. **Question 3:** How many different sums are possible?

Looking Back Instead of four regions, suppose the dartboard had three regions. **Question 4:** How many different sums would be possible on a dartboard with three regions numbered 1, 5, and 10?

Answers to Questions 1–4 **1.** The largest sum is 90, and the smallest is 3. **2.** The possible sums are 7, 11, 16, and 36. **3.** 20 different sums. **4.** 10 different sums.

HISTORICAL HIGHLIGHT



This adding machine was developed by Blaise Pascal in 1642 for computing sums. The machine is operated by dialing a series of wheels with digits from 0 to 9. To carry a number to the next column when a sum is greater than 9, Pascal devised a ratchet mechanism that would advance a wheel 1 digit when the wheel to its right made a complete revolution. The wheels from right to left represent units, tens, hundreds, etc.