# 3

Whole Numbers

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Grades Pre-K through 2*

Teachers have a very important role to play in helping students develop facility with computation. By allowing students to work in ways that have meaning for them, teachers can gain insight into students' developing understanding and give them guidance. . . . Consider the following hypothetical story, in which a teacher poses this problem to a class of second graders:

> We have 153 students at our school. There are 273 students at the school down the street. How many students are in both schools?
>
> As would be expected in most classrooms, the students give a variety of responses that illustrate a range of understandings. For example, Randy models the problem with bean sticks that the class has made earlier in the year, using hundreds rafts, tens sticks, and loose beans.
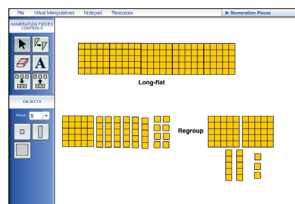

12 tens

3 rafts

6 beans
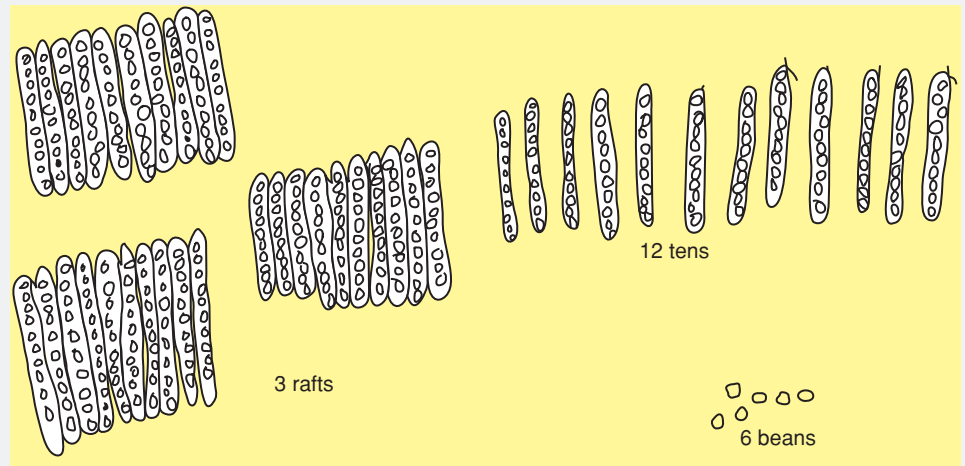
*_Principles and Standards for School Mathematics_, p. 86.

MATH ACTIVITY **3.1**

Numeration and Place Value with Base-Five Pieces

Purpose: Explore whole number numeration concepts with base-five pieces. Base-five pieces are used to provide a fresh look at numeration concepts and to help develop a deeper understanding of numeration systems.

Materials: Base-Five Pieces in the Manipulative Kit or Virtual Manipulatives.

1. The four base-five pieces shown here are called **unit, long, flat,** and **long-flat.** Examine the numerical and geometric patterns of these pieces as they increase in size, and describe how you would design the next larger piece to continue your pattern.

Long-flat   Flat   Long   Unit
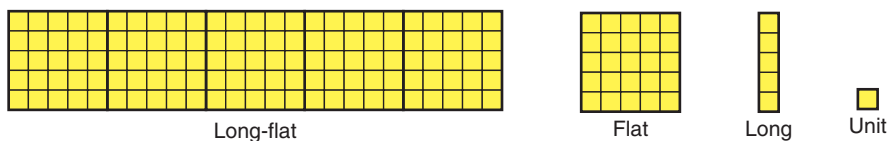
*2. The two collections shown here both contain 36 units, but collection 2 is called the **minimal collection** because it contains the smallest possible number of base-five pieces. Use your base-five pieces to determine the minimal collection for each of the following numbers of units.

Collection 1   Collection 2
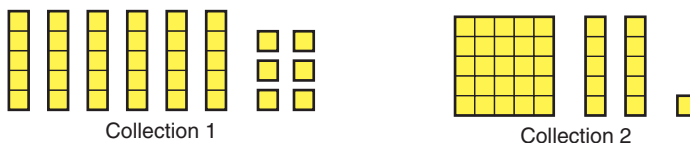
	Long-Flats	Flats	Longs	Units
a. 84 units				
b. 147 units				
c. 267 units				

3. In base-five numeration, 3 flats, 2 longs, and 4 units are recorded by the numeral 324_{five}. Sketch the base pieces for each of the following numerals, and determine the total number of units in each collection.

   a. 1304_{five} **b.** 221_{five} **c.** 213_{five} **d.** 1023_{five}

*4. Starting with the unit, sketch the first four base-three pieces.

   a. What is the minimal collection of base-three pieces for 16 units?

   b. What is the total number of units represented by 2112_{three} (2 long-flats, 1 flat, 1 long, 2 units)?

Section **3.1**   NUMERATION SYSTEMS

Egyptian stone giving an account of the expedition of Amenhotep III in 1450 B.C.E.
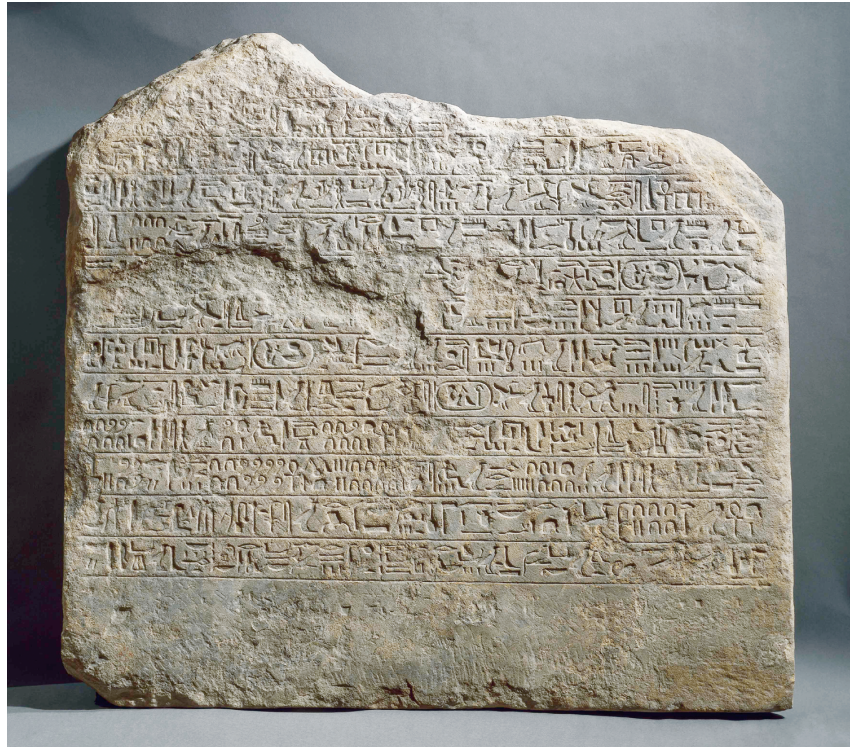
PROBLEM OPENER

A 7 is written at the right end of a two-digit number, thereby increasing the value of the number by 700. Find the original two-digit number.

There are no historical records of the first uses of numbers, their names, and their symbols. A number is an idea or abstraction that represents a quantity. Written symbols for numbers are called **numerals** and probably were developed before number words, since it is easier to cut notches in a stick than to establish phrases to identify a number.

A logically organized collection of numerals is called a **numeration system.** Early numeration systems appear to have grown from tallying. In many of these systems, 1, 2, and 3 were represented by **❙**, **❙❙**, and **❙❙❙**. By 3400 B.C.E. the Egyptians had an advanced system of numeration for numbers up to and exceeding 1 million. Their first few number symbols show the influence of the simple tally strokes (Figure 3.1).

Figure 3.1

Their symbol for 3 can be seen in the third row from the bottom of the stone inscriptions shown above. What other symbols for single-digit numerals can you see on this stone?

GROUPING AND NUMBER BASES

As soon as it became necessary to count large numbers of objects, the counting process was extended by grouping. Since the fingers furnished a convenient counting device, grouping by 5s was used in some of the oldest methods of counting. The left hand was generally used to keep a record of the number of objects being counted, while the right index finger pointed to the objects. When all 5 fingers had been used, the same hand would be used again to continue counting. In certain parts of South America and Africa, it is still customary to "count by hands": 1, 2, 3, 4, hand, hand and 1, hand and 2, hand and 3, etc.

EXAMPLE A

Use the "count by hands" system to determine the names of the numbers for each of the following sets of dots.

1. • • • • • • •
   • • • • • • •

2. • • • • • • •
   • • • • • • •
   • • • • • •

3. • • • • • • •
   • • • • • • •
   • • • • • • •
   • • • • •

Solution   **1.** 2 hands and 2.   **2.** 3 hands and 4.   **3.** 4 hands and 3.

The number of objects used in the grouping process is called the **base.** In Example A the base is five. By using the numerals 1, 2, 3, and 4 for the first four whole numbers and *hand* for the name of the base, it is possible to name numbers up to and including 24 (4 hands and 4).

Base Ten   As soon as people grew accustomed to counting by the fingers on one hand, it became natural to use the fingers on both hands to group by 10s. In most numeration systems today, grouping is done by 10s. The names of our numbers reflect this grouping process. *Eleven* derives from the medieval German phrase *ein lifon,* meaning one *left over,* and *twelve* is from *twe lif,* meaning *two over ten.* The number names from 13 to 19 have similar derivations. *Twenty* is from *twe-tig,* meaning *two tens,* and *hundred* means *ten times ten.** When grouping is done by 10s, the system is called a **base-ten numeration system.**

ANCIENT NUMERATION SYSTEMS

Egyptian Numeration   The ancient Egyptian numeration system used picture symbols called **hieroglyphics** (Figure 3.2). This is a base-ten system in which each symbol represents a power of ten.

HISTORICAL HIGHLIGHT

There are many traces of base twenty from different cultures (see Example F). The Mayas of Yucatán and the Aztecs of Mexico had elaborate number systems based on 20. Greenlanders used the expression *one man* for twenty, *two men* for 40, etc. A similar system was used in New Guinea.

Evidence of grouping by 20 among the ancient Celtics can be seen in the French use of *quatre-vingt* (four-twenty) for 80. In our language the use of *score* suggests past tendencies to count by 20s. Lincoln's familiar Gettysburg Address begins, "Four score and seven years ago." Another example occurs in a childhood nursery rhyme: "Four and twenty blackbirds baked in a pie."

*H. W. Eves, *An Introduction to the History of Mathematics,* 3d ed. (New York: Holt, Rinehart and Winston, 1969), pp. 8–9.

1,000,000	100,000	10,000	1000	100	10	1
Astonished man	Tadpole	Pointing finger	Lotus flower	Coiled rope	Heel bone	Stick

Egyptian Symbols

Figure 3.2

EXAMPLE B

Write the following numbers, using Egyptian numerals.

1. 2342   **2.** 14,026

Solution   1.   2.

The Egyptian numeration system is an example of an *additive numeration system* because each power of the base is repeated as many times as needed.

> **Additive Numeration System** In an **additive numeration system,** some number b is selected for a base and symbols representing 1, b, b^2, b^3, etc., for powers of the base. Numbers are written by repeating these powers of the base the necessary number of times.

In the Egyptian numeration system, $b = 10$ and the powers of the base are 1, 10, 10^2, 10^3, etc. In an additive numeration system, the symbols can be written in any order. In Example B the powers of the base are descending from left to right, but the Egyptian custom was to write them in ascending powers from left to right, as shown in the stone inscriptions on page 125.

EXAMPLE C

Notice the numeral for 743 near the left end of the third row from the bottom of the Egyptian stone on page 125. The symbols for 3 ones, 4 tens, and 7 hundreds are written from left to right. What other Egyptian numerals can you find on this stone?

Solution   It appears that the center of the third row up contains 75 and the left end of the fourth row up has 250. Parts of many other numerals can be seen.

Roman Numeration Roman numerals can be found on clock faces, buildings, gravestones, and the preface pages of books. Like the Egyptians, the Romans used base ten. They had a modified additive numeration system, because in addition to the symbols for powers of the base, there are symbols for 5, 50, and 500. The seven common symbols are

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

Roman Numerals

Historical evidence indicates that C is from *centum,* meaning *hundred,* and M is from *milli,* meaning *thousand.* The origin of the other symbols is uncertain. The Romans wrote their numerals so that the numbers they represented were in decreasing order from left to right.

EXAMPLE D

Write the following numbers, using Roman numerals.

1. 2342   **2.** 1996

Solution   **1.** MMCCCXXXXII   **2.** MDCCCCLXXXXVI

When a Roman numeral is placed to the left of a numeral for a larger number, its position indicates subtraction, as in IX for 9, XL for 40, XC for 90, CD for 400, or CM for 900. The subtractive principle was recognized by the Romans, but they did not make much use of it.* (In fact, the subtractive principle has only been in common use for about the past 200 years.) Compare the preceding Roman numeral for 1996 with the following numeral written using the subtractive principle:

<div align="center">MCMXCVI</div>

The Romans had relatively little need for large numbers, so they developed no general system for writing them. In the inscription on a monument commemorating the victory over the Carthaginians in 260 B.C.E., the symbol ⌒ for 100,000 is repeated 23 times to represent 2,300,000.

Babylonian Numeration The Babylonians developed a base-sixty numeration system. Their basic symbols for 1 through 59 were additively formed by repeating ⟁ for 1 and ◀ for 10. Four such numerals are shown here.

23	6	40	59
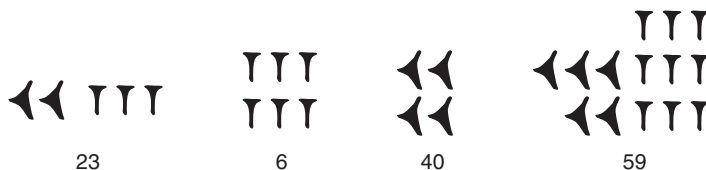
To write numbers greater than 59, the Babylonians used their basic symbols for 1–59 and the concept of *place value*. **Place value** is a power of the base, and the Babylonian place values were 1, 60, 60^2, 60^3, etc. Their basic symbols had different values depending on the position or location of the symbol. For example, $135 = 2(60) + 15(1)$, so the Babylonians wrote their numeral for 2 to represent 2×60 and their numeral for 15 for the number of units, as shown next. Generally, the first position from right to left represented the number of units, the second position the number of 60s, the third position the number of 60^2s, etc.

$$2(60) + 15(1) = 135 \qquad 22(60^2) + 3(60) + 30(1) = 79,410$$

EXAMPLE E

Write the following numbers using Babylonian numeration.

1. 47   **2.** 2473   **3.** 10,821

Solution   **1.**   **2.**   **3.**

$$47(1) \qquad\qquad 41(60) + 13(1) \qquad\qquad 3(60^2) + 21(1)$$

*D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn 1925), p. 60.

The solution to the third part of Example E illustrates a weakness in the Babylonian system. The number 10,821 is equal to $3(60^2) + 0(60) + 21(1)$, but because there was no symbol for zero in the Babylonian system, there was no way to indicate 0(60), that is, the missing power of 60. Babylonians who saw the symbols in the third part of Example E might have thought it represented $3(60) + 21(1)$. A larger gap between symbols was sometimes used to indicate that a power of the base was missing, and later, symbols were used to indicate a missing power of the base.

Mayan Numeration   The Mayas used a modified base-twenty numeration system that included a symbol for zero. Their basic symbols for 0 through 19 are shown in Figure 3.3. Notice that there is grouping by 5s within the first 20 numbers.
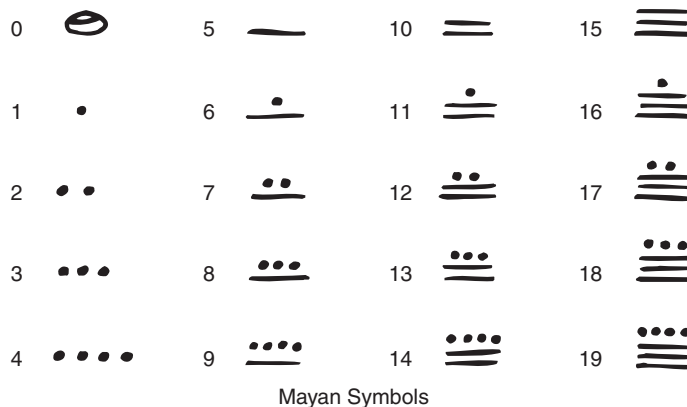

Figure 3.3

Mayan Symbols

To write numbers greater than 19, the Mayas used their basic symbols from 0 to 19 and place value. They wrote their numerals vertically with one numeral above another, as shown in Example F, with the powers of the base increasing from bottom to top. The numeral in the bottom position represented the number of units. The numeral in the second position represented the number of 20s. Because the Mayan calendar had 18 months of 20 days each, the place value of the third position was 18×20 rather than 20^2. Above this position, the next place values were 18×20^2, 18×20^3, etc.

EXAMPLE F

These three Mayan numerals represent the following numbers: $16(20) + 6(1) = 326$; $7(18 \times 20) + 12(20) + 16(1) = 2776$; and $9(18 \times 20^2) + 2(18 \times 20) + 0(20) + 6(1) = 65,526$.

326   2776   365,525
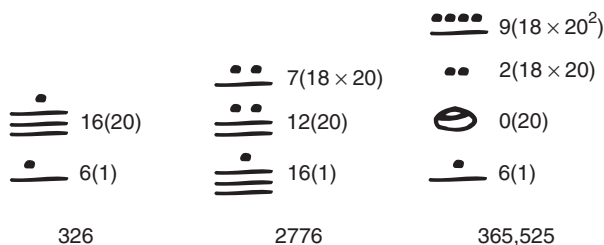
EXAMPLE G

Write the following numbers using Mayan numerals.

1. 60   **2.** 106   **3.** 2782

Solution   **1.** ••• 3(20)   **2.** ── 5(20)   **3.** •• $7(18 \times 20)$
            ⬯ 0(1)   • 6(1)   ••• 13(20)
                                                  •• 2(1)

Notice the necessity in the Mayan system for a symbol that has the same purpose as our numeral zero. In part 1 of Example G, their symbol for zero occupies the lower place and tells us that the three dots have a value of 3×20 and there are zero 1s.

HISTORICAL HIGHLIGHT

There is archaeological evidence that the Mayas were in Central America before 1000 B.C.E. During the Classical Period (300 to 900), they had a highly developed knowledge of astronomy and a 365-day calendar with a cycle going back to 3114 B.C.E. The pyramid of Kukulkan at Chichén Itzá, pictured at the left, was used as a calendar: four stairways, each with 91 steps and a platform at the top, made a total of 365. Their year was divided into 18 months of 20 days each with 5 extra days for holidays. Because their numeration system was developed mainly for calendar calculations, they used 18×20 for the place value in the third position, rather than 20×20.

Technology Connection

Can you decipher this Attic-Greek ancient numeration system? To decipher this system, drag the symbols onto the workspace and click "Translate Symbols" to determine their value. There are five other ancient numeration systems for you to decipher in this applet.

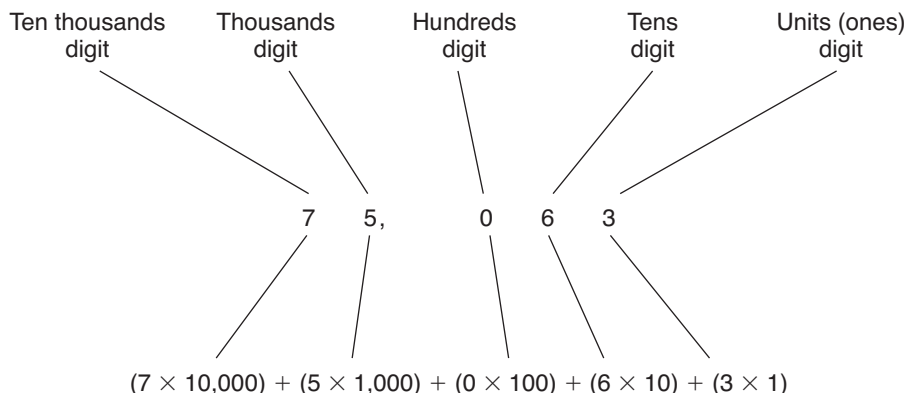Deciphering Ancient Numeration Systems Applet, Chapter 3
www.mhhe.com/bbn

Hindu-Arabic Numeration Much of the world now uses the Hindu-Arabic numeration system. This positional numeration system was named for the Hindus, who invented it, and the Arabs, who transmitted it to Europe. It is a base-ten numeration system in which place value is determined by the position of the **digits** 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Each digit in a numeral has a name that indicates its position.

EXAMPLE H

Here are the names and values of the digits in 75,063.

Ten thousands digit	Thousands digit	Hundreds digit	Tens digit	Units (ones) digit

$$7 \quad 5, \quad 0 \quad 6 \quad 3$$

$$(7 \times 10{,}000) + (5 \times 1{,}000) + (0 \times 100) + (6 \times 10) + (3 \times 1)$$

When we write a number as the sum of the numbers represented by each digit in its numeral (see Example H), we are writing the number in **expanded form.** Another common method of writing a number in expanded form is to write the powers of the base using exponents. For example, $7(10^4) + 5(10^3) + 0(10^2) + 6(10^1) + 3(1)$.

The Hindu-Arabic numeration system is an example of a positional numeration system. In general,

> **Positional Numeration System** In a **positional numeration system,** a number is selected for a base and basic symbols are adopted for 0, 1, 2, . . . up to one less than the base. (In our numeration system these basic symbols are the 10 digits 0, 1, 2, . . . , 9.) Whole numbers are represented in a positional numeration system by writing one or more basic symbols side by side with their positions indicating increasing powers of the base.

EXAMPLE I

Determine the value of each underlined digit and its place value.

  1. 7024   **2.** 370,189   **3.** 49,238

Solution   **1.** The value is 0, and the place value is hundreds.   **2.** The value is 70,000, and the place value is ten thousands.   **3.** The value is 200, and the place value is hundreds.

READING AND WRITING NUMBERS

In English the number names for the whole numbers from 1 to 20 are all single words. The names for the numbers from 21 to 99, with the exceptions of 30, 40, 50, etc., are compound number names that are hyphenated. These names are hyphenated even when they occur as parts of other names. For example, we write *three hundred forty-seven* for 347.

Numbers with more than three digits are read by naming each group of three digits (the **period** of the digits). Within each period, the digits are read as we would read any number from 1 to 999, and then the name of the period is recited. The names for the first few periods are shown in the following example.

EXAMPLE J   Read the following number.

$$\underbrace{2\ \ 3,}_{\text{Trillion}}\ \ \underbrace{4\ \ 7\ \ 8,}_{\text{Billion}}\ \ \underbrace{5\ \ 0\ \ 6,}_{\text{Million}}\ \ \underbrace{0\ \ 4\ \ 2,}_{\text{Thousand}}\ \ 3\ \ 1\ \ 9$$

Solution   This number is read as twenty-three trillion, four hundred seventy-eight billion, five hundred six million, forty-two thousand, three hundred nineteen. *Note:* The word *and* is not used in reading a whole number.

HISTORICAL HIGHLIGHT
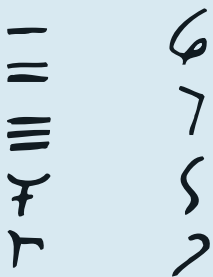
There are various theories about the origin of our digits. It is widely accepted, however, that they originated in India. Notice the resemblance of the Brahmi numerals for 6, 7, 8, and 9 to our numerals. The Brahmi numerals for 1, 2, 4, 6, 7, and 9 were found on stone columns in a cave in Bombay dating from the second or third century B.C.E.* The oldest dated European manuscript that contains our numerals was written in Spain in 976. In 1299, merchants in Florence were forbidden to use these numerals. Gradually, over a period of centuries, the Hindu-Arabic numeration system replaced the more cumbersome Roman numeration system.

*J. R. Newman, *The World of Mathematics* (New York: Simon and Schuster, 1956), pp. 452–454.

ROUNDING NUMBERS

If you were to ask a question such as "How many people voted in the 2008 presidential election?" you might be told that in "round numbers" it was about 131 million. Approximations are often as helpful as the exact number, which in this example is 131,257,328.

One method of **rounding** a number to the nearest million is to write the nearest million greater than the number and the nearest million less than the number and then choose the closer number. Of the following numbers, 131,257,328 is closer to 131,000,000.

                        132,000,000
                        131,257,328 rounds to 131,000,000
                        ⟶
                        131,000,000

The more familiar approach to rounding a number uses place value and is stated in the following rule.

Rule for Rounding Whole Numbers

1. Locate the digit with the place value to which the number is to be rounded, and check the digit to its right.

2. If the digit to the right is 5 or greater, then each digit to the right is replaced by 0 and the digit with the given place value is increased by 1.

3. If the digit to the right is 4 or less, each digit to the right of the digit with the given place value is replaced by 0.

EXAMPLE K

Round 131,257,328 to the following place values.

1. Ten thousands   **2.** Thousands   **3.** Hundreds

Solution   **1.** Ten thousands place
↓
131,257,328 rounds to 131,260,000
⟶

2. Thousands place
↓
131,257,328 rounds to 131,257,000
⟶

3. Hundreds place
↓
131,257,328 rounds to 131,257,300
⟶

MODELS FOR NUMERATION

NCTM's K–4 Standard, *Number Sense and Numeration* in the *Curriculum and Evaluation Standards for School Mathematics* (p. 39), says that place value is a critical step in the development of children's understanding of number concepts:

> Since place-value meanings grow out of grouping experiences, counting knowledge should be integrated with meanings based on grouping. Children are then able to use and make sense of procedures for comparing, ordering, rounding, and operating with larger numbers.

There are many models for illustrating positional numeration and place value. The bundles-of-sticks model and base-ten number pieces will be introduced in Examples L, M, and N and then used to model operations on whole numbers in the remainder of this chapter.

Bundles-of-Sticks (or Straws) Model In this model, units and tens are represented by single sticks and bundles of 10 sticks, respectively. One hundred is represented by a bundle of 10 bundles.
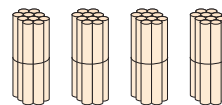
EXAMPLE L

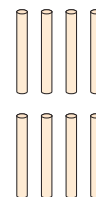The following figure shows the bundle-of-sticks model for representing 148.

Using concrete materials can help students learn to group and ungroup by tens. For example, . . . to express "23" as 23 ones (units), 1 ten and 13 ones, or 2 tens and 3 ones. p. 81
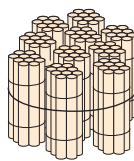
100   40   8

Base-Ten Pieces In this model, the powers of 10 are represented by objects called **units, longs,** and **flats:** 10 units form a long, and 10 longs form a flat (Figure 3.4). Higher powers of the base can be represented by sets of flats. For example, 10 flats placed in a row are called a **long-flat** and represent 1000.
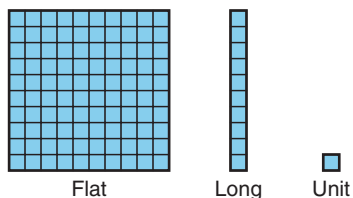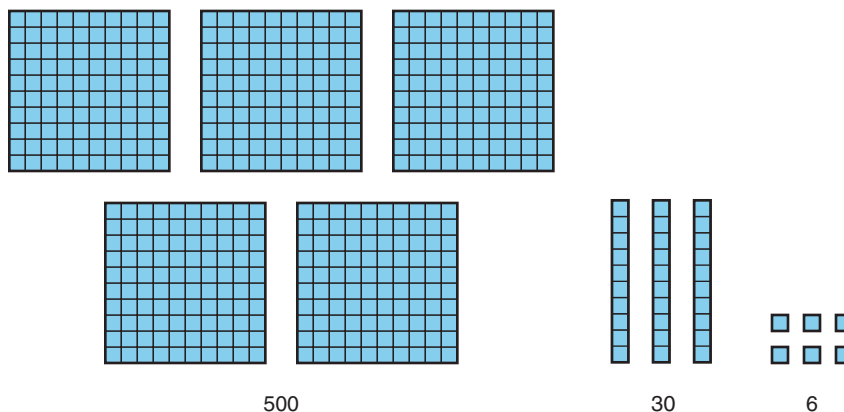

Figure 3.4

Flat   Long   Unit

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EXAMPLE M   Sketch base-ten pieces to represent 536.

Solution

500   30   6

Bundles of sticks and base-ten pieces can be used to illustrate the concept of **regrouping:** changing one collection to another that represents the same number.

EXAMPLE N   Sketch the minimum number of base-ten pieces needed to replace the following collection. Then determine the base-ten number represented by the collection.
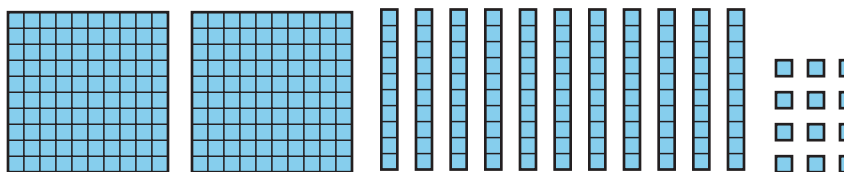
Solution   The new collection will have 3 flats, 2 longs, and 2 units. This collection of base-ten pieces represents 322.

Base-Five Numeration The base-five pieces are models for powers of 5, and as in the case of base ten, there are pieces called **units, longs,** and **flats:** 5 units form a long, and 5 longs form a flat. The next higher power of 5 is represented by placing 5 flats end to end to form a **long-flat.**
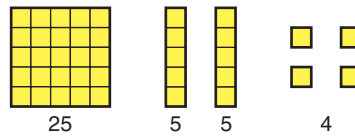
EXAMPLE O

Sketch the minimum number of base-five pieces to represent the following number of units.

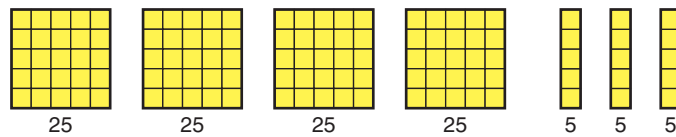**1.** 39 units   **2.** 115 units   **3.** 327 units
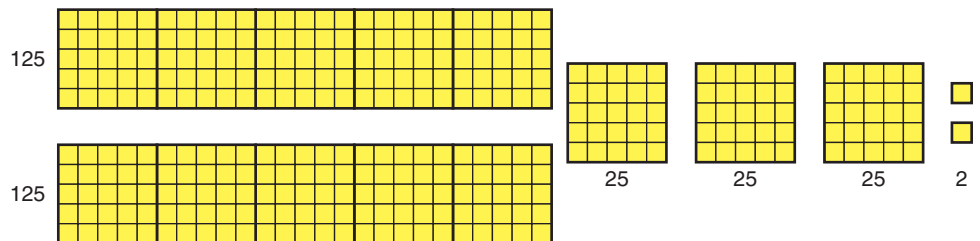
Solution   **1.** A collection with 1 flat, 2 longs, and 4 units

2. A collection with 4 flats, 3 longs, and 0 units

**3.** A collection with 2 long-flats, 3 flats, 0 longs, and 2 units

Positional numeration is used to write numbers in various bases by writing the numbers of long-flats, flats, longs, and units from left to right, just as we do in base ten. From Example O, 39 in base-ten numeration is written as 124_{five} in base-five numeration. Similarly, 115 is written as 430_{five}, and 327 is written as 2302_{five}. Since base ten is the standard base, we do not write the subscript to show that a number is being written in base-ten numeration.

PROBLEM-SOLVING APPLICATION

The next problem introduces the strategy of **reasoning by analogy,** which involves forming conclusions based on similar situations. For example, we know that when we add two numbers, *the greater the numbers, the greater the sum.* Reasoning by analogy, we might conclude that *the greater the numbers, the greater the product.* In this case the conclusion is true. This type of reasoning, however, is not always reliable; the conclusion *the greater the numbers, the greater the difference* would be false. The problem-solving strategies of *reasoning by analogy* and *using a model* are used on the next page to solve a problem involving base-twelve positional numeration.

Problem

Flat   Long   Unit
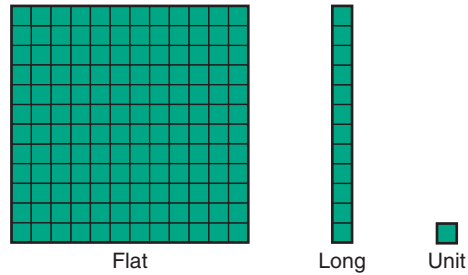
How can numbers be written in base-twelve positional numeration?

Understanding the Problem The count-by-hands method of counting, which was introduced in the opening pages of this section, is a base-five system. **Question 1:** In that system, what digits are needed to name any number from 1 to 24?

Devising a Plan Consider a similar problem: Why are 0, 1, 2, 3, . . . , 9 the only digits needed in base ten? Referring to the base-ten pieces, we know that if there are more than nine of one type of base-ten piece, we can replace each group of 10 pieces by a piece representing the next higher power of 10. This suggests using similar pieces for base twelve. The first three base-twelve pieces are shown above. **Question 2:** How can this model be extended?

Carrying Out the Plan To count in base twelve, we can say 1, 2, 3, 4, 5, 6, 7, 8, 9, but then we need new symbols for ten and eleven because 10 in base twelve represents 1 long and 0 units, which equals twelve units; and 11 in base twelve represents 1 long and 1 unit, which equals thirteen units. One solution is to let T represent the number 10 and E represent the number eleven. Then the first few numerals in base twelve are 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, 13, . . . , where 12 represents 1 long and 2 units (fourteen units), etc. In base-twelve positional numeration, 3 flats, 2 longs, and 8 units are written as 328_{twelve}. **Question 3:** Why does any base-twelve numeral require only the twelve symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E?

Looking Back These models suggest ways to visualize other number bases, such as base two, base seven, or base sixteen. **Question 4:** What digits are needed in base two, and what would the base-two pieces look like?

Answers to Questions 1–4   1. 0, 1, 2, 3, and 4. For example, 2 hands and 1, 3 hands and 4, etc. **2.** The next base-twelve piece has a row of 12 flats.   **3.** Whenever there are 12 of any base-twelve pieces, they can be replaced by the next larger base-twelve piece. If there are no pieces of a given type, the 0 is needed in the numeral to indicate this.   **4.** The only digits needed in base two are 0 and 1. The first four base-two pieces are shown here.

Long-flat   Flat   Long   Unit
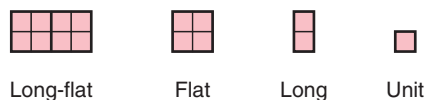
Technology Connection

One skill in helping children acquire number sense and familiarity with place value is counting up by 10s from a given number. This can be practiced with a calculator that has a constant function key or is programmed to achieve this function. On such calculators, which are designed for elementary school students, the following key strokes will produce a sequence of numbers, with each number being 10 more than the preceding number.

Keystrokes	View Screen
26	26
+ 10 =	36
=	46
=	56

In a similar manner, but using ⊟, a calculator with a constant function enables students to practice counting down by 10s. If a calculator does not have a constant function, the preceding sequence 26, 36, 46, . . . can be generated on most calculators by entering 26 and repeatedly pressing ⊞ 10 ⊟. Similarly, the keystrokes below will produce the decreasing sequence 54, 44, 34, ...

Keystrokes	View Screen
64 − 10 =	54
− 10 =	44
− 10 =	34
− 10 =	24

HISTORICAL HIGHLIGHT

In the fifteenth and sixteenth centuries, there were two opposing opinions on the best numeration system and methods of computing. The *abacists* used Roman numerals and computed on the abacus and the *algorists* used the Hindu-Arabic numerals and place value. The sixteenth-century print at the left shows an abacist competing against an algorist. The abacist is seated at a reckoning table with four horizontal lines and a vertical line down the middle. Counters, or chips, placed on lines represented powers of 10. The thousands line was marked with a cross to aid the eye in reading numbers. If more lines were needed, every third line was marked with a cross. This practice gave rise to our modern custom of separating groups of three digits in a numeral by a comma.*

*D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), pp. 183–185.