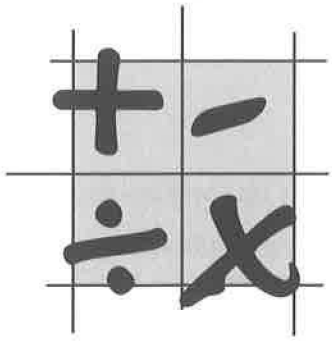


CHAPTER

3

Written numerals and the structure of tens and ones

In chapter 3, we continue the investigation of children's understandings of how numbers are decomposed, but take another step back for a wider perspective. Rather than examine children who are working on computation, we consider children who are learning about how the components of a number relate to written numerals. It is easy for adults to see, for example, that the numeral 706 represents 7 hundreds, 0 tens, and 6 ones. However, when we study children who are in the process of learning, we discover that this idea is not so simple.

As you read chapter 3, take notes on these issues:

- In the following cases, some of the children offer incorrect answers. Look for elements of logic in their thinking. If the children are working from a logical position, where does their thinking go awry?
- In other cases, the children are talking about their own new insights. What are these insights and how do they come about?

Although the cases present the thinking of children in primary grades, teachers of older children can gain insight into their students' thinking as well. In subsequent chapters, we will see children sorting out these same ideas in the contexts of multiplying and dividing multidigit numbers and learning to work with decimals.

CASE 11

Number of days in school

Dawn

KINDERGARTEN, DECEMBER

Each morning, as my kindergarten class gathers on the meeting rug, we run through a routine that helps set our day in motion. This set of rituals includes taking attendance, working with our classroom calendar and weather graph, and recording how many days we have been in school. It is amazing how much mathematics is involved in these activities, and often I am astounded by the thoughtful responses five- and six-year-olds give to such questions as, "What number should we record on our days-in-school chart today?" This morning, our sixtieth day of school, we turned to our days-in-school chart.

Over the years, I have spent a lot of time thinking about the optimal way to record this data with kindergarten students. Many years ago I used a number line that spanned the top of the chalkboard, but found that this was too physically removed from the children, not to mention extremely cumbersome for me. Having seen in the past the merits of

using hundreds boards when working with older students, I wondered if this type of grid might have a place in the kindergarten classroom as well. About ten years ago I made a switch to this type of recording system for tracking how many days we have been in school each year. Every day I record the number on our 10-by-18 grid, and the child who is the calendar-helper adds one seashell to a cup that we keep nearby. This provides a set of concrete objects that corresponds to the number being logged on the chart. From time to time, the children count this set of shells, and we then compare our two types of recording systems to connect the quantity of shells with the number we count, read, and write. If necessary, we then adjust our data.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	

To begin our discussion this morning, I asked, "What number should I write on our chart today?" Hands shot up and I began writing responses on the chalkboard next to our chart, hoping to allow as many children as possible to respond before recording the "right answer" on the grid. Often this type of discussion yields some interesting discoveries. Today was no exception.

ANDREW: I think it's fifty-ten. [*I wrote 510 on the board.*]

JOSEF: Sixty. [*I wrote 60.*]

Dawn

KINDERGARTEN, DECEMBER

More hands shot up, and Bianca, Jared, Rhea, Terry, Toshi, Pat, John, Sione, Susan, and Brady all responded "Sixty." Some children added other comments, too.

TOSHI: I know it's 60. I just know it is.

Jared, John, and Sione were equally emphatic. Susan seemed a little less sure, but apparently wanted to go along with the general consensus. Still more responses kept coming as more hands were up.

TAMIKA: Forty. [*I wrote 40.*]

JERREL: Eight. [*I wrote 8.*]

NINA: Seventy. [*I wrote 70.*]

"How can we find out which number we should write today?" I asked, to continue our discussion.

BIANCA: Counting, so we could know what comes next.

JOSEF: All the numbers are in front of it, 'cause 6 comes after 5, you know.

As Josef spoke, he moved up to the chart and pointed to the column of numbers on the right hand side, stopping at the empty box under the 50.

JARED: Yeah, see, zero all the way down.

ANDREW: Don't forget the 10.

Once again, physical involvement seemed a necessity as Andrew moved up to the chart and dragged his finger across the row with the numbers 51, 52, 53, 54, 55, 56, 57, 58, and 59. Andrew seemed to be making use of the number (counting) sequence; thus his response of "fifty-ten" (510) made perfect sense.

BRADY: But it's a 6. [*Again, moving up to point at the chart, Brady's finger slid down the right hand column just as Toshi's and Jared's had as they spoke.*] See, 1, 2, 3, 4, 5, and 6 goes here 'cause you're counting down.

TOSHI: See, it's 60. I know it is.

Sitting toward the back of the group was Norman. Though he didn't speak out in front of the group, my aide was taking notes and later related to me that he was making comments under his breath.

NORMAN: It can't be 8. We already had that one. . . . Five and ten looks like five hundred and ten. That is too big to go there.

Because we needed to move on, I ended this discussion by reminding the children that another way to check was to use Bianca's suggested strategy of counting. Together, as I pointed to each number on our chart, we counted from 1 to 60 and agreed that 60 was the number to be written in the box for today. As we finished, there were two final comments:

JOSEF: Are we going to get 100?

BRADY: Yes, 'cause look, we can count by tens: 10, 20, 30, 40, 50, 60. I think we will get to more than 100.

As I reflect on these events, I am particularly taken by the ease with which some children are able to connect to the systematic way we use numbers. At the same time, I know for sure that not all children in my class are making sense of this experience at this time. Trying to provide opportunities where young children can investigate numbers in a meaningful way is a challenge.

CASE 13

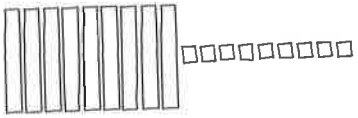
Ones, tens, hundreds

Marie

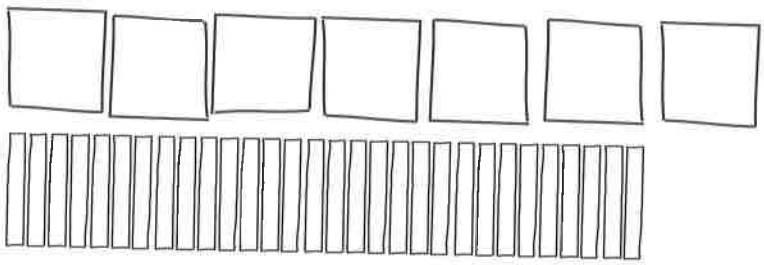
GRADE 3, MARCH

We have been working on place value all year, while doing activities that range from counting on a hundreds chart to multiplying three-digit by one-digit numbers. In this lesson, my objectives were to revisit the writing of numbers, focusing on the value of the digits in the hundreds, tens, and ones places by using expanded notation, and representing these quantities with base ten blocks and by drawing pictures. Most of the children can write numbers. The confusion lies in representing the amounts. Here are some examples:

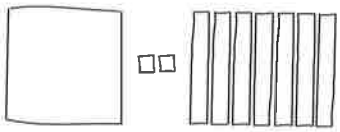
Lou showed 99 by laying out 9 rods (ten-sticks) and 9 units. His written number sentence to represent this amount was 99 = 9 + 9. His picture looked like this:



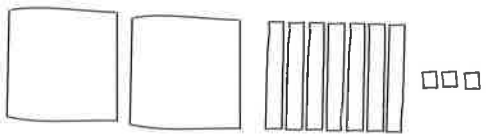
José wrote 730 = 70 + 30 + 0. He laid out 7 flats (100-squares) and 30 rods to show this quantity. Then he drew:



Mary showed 127 as one flat, 2 units, and 7 rods, which she arranged in the same order, as her picture shows. Her number sentence was 127 = 100 + 27.



In another session I had the chance to talk with some of the children. Anne, Tia, Lou, James, Lee, and Darryl stayed with me while the rest of the class went to another activity. I gave out the papers and base ten blocks, and then gave each student a number to write down. The students each used the blocks to build their number, and then drew a picture of what they had built. Darryl's number was 237, which he showed by lining up 2 flats, 7 rods, and 3 units, in this order:



I asked him how he would form a group of 10 units.

DARRYL: Take 10 ones and stick them together to make a group of 10.

TEACHER: Lou, do you understand what Darryl said?

LOU: If he breaks up a 10 into 10 ones, it will still be 10.

Marie

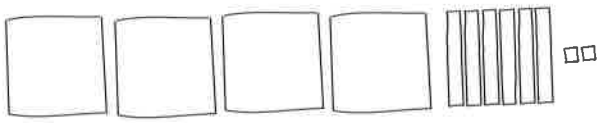
GRADE 3, MARCH

Next, students picked their own number to represent. They made the same mistakes as before.

TEACHER: What is your number, Tia?

TIA: It's 426.

She laid out 4 flats, 6 rods, and 2 units, as shown.



TIA: [*Pointing to each block as she counts from left to right*] 100, 200, 300, 400, 401, 402 . . .

TEACHER: Why count the ones before the tens?

TIA: 400, 10, 20, 30, 40, 50, 60.

TEACHER: Where is the 60? [*Tia points to the 6 in 426.*] What does the 4 stand for?

TIA: 400.

TEACHER: The 2?

TIA: It stands for the ones.

I wrote 426 and circled the 26. Then I asked Tia what the number was, and she answered correctly.

CASE 14

Who invented zero anyway?

Muriel

GRADE 2, APRIL

My second graders and I were looking at the hundreds chart set up in a 10-by-10 array. I had imagined leading a discussion toward the idea that moving down one space is actually adding 10. As we got into the discussion, I found that it is, of course, a complicated idea; even if I just tell my students that it works this way, they don't "get it" in any kind of meaningful way. Besides that, several of my students raised a very different idea—an idea about their understanding of zero. Following is a portion of our discussion.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Beth is describing for me—and anyone who is listening—why it works to move down one and actually add ten on this particular hundreds chart.

Muriel

GRADE 2, APRIL

BETH: See the nines? [*She points to the column of 9, 19, 29, 39 . . .*] This is 0 + 9, 10 + 9, 20 + 9, 30 + 9. The difference between 50 and 60 is 10, so the difference between 59 and 69 is 10.

Quite articulate, I think. Then Beth continues.

BETH: [*She points to 45.*] This has 4 tens and one 5. These [*pointing to the columns with numbers ending in 6 or 7*] all have sixes, sevens. [*Now she points to the 10, 20, 30 column.*] This one has zeros. But these aren't quite zero, because. . . .

Her voice kind of trails off and my mind is racing. What did she say? "These aren't quite zero"?

TEACHER: Because why? [*I am trying desperately to figure out in that moment all of what she is saying and trying to sort out what I might ask next.*]

BETH: Because if zero isn't here [*she points to 60*], then this 6 is only 6. It depends on where it [the zero] is. See, 15. [*She writes 15 on the board.*] This [*points to the 1*] is a 10, not a 1. Ten has everything in it up to 9. The ten section has from 10 to 99. The hundred section has—say it's one hundred and twenty-five. [*She writes 125 on the board.*] But you don't write it as 100 or else it would look like this. [*She writes 100205 on the board.*]

I'm in that place where I find myself spending a lot of time: hearing many rich ideas, and wondering which ideas I should push on. I decide to push on Beth's ideas about the zero thing.

TEACHER: You've written 125 separately, as 100205. But tell me some more about zero.

BETH: It is sort of zero, but not exactly. [In 30] this zero makes it be 30. If this zero weren't there, it'd be 3.

YESSICA: I have something to say about how these zeros aren't really zeros.

Attempting to include some other children in the discussion, I restate what Beth has said, about the zero at the end of the number not being quite the same as zero by itself. Yessica comes up to the board.

YESSICA: [*Writing 07 on the board*] That's 7. [*Now writes 0.7*] That's 0 point 7. [*And then she writes 70.*] That's 70. Zero represents 7 tens.

I am completely intrigued by these ideas and love the term *represents.*

TEACHER: So if this zero [*pointing to 0 in 70*] represents tens, in this number, 79, does the 9 represent tens?

BETH: Do you know what we really mean? Do you know the real thing?

Lately Beth has been responding to my questions as if I did already know many of the answers I'm asking her to explain to me. I like to make my questioning as authentic as I can, but the fact is that I usually understand the mathematics that I'm asking the children to think through. Today my questions are framed to find out what the children are thinking, to hear their ideas. I know that the 9 in 79 represents 9 ones, but I'm not sure what Beth thinks.

I explain to her that in my math class for adults, we've talked a lot about zero and what it means to different people, and what it's worth, and whether it's odd or even, for example. And I express my honest interest in what second graders think about zero. This is heard by several other people in class and they perk up a bit. The discussion continues.

YESSICA: On the calculator there's a 07.

TEACHER: And then is the 0 worth zero?

YESSICA: Yes. [*Other children nod in agreement.*]

TEACHER: But not in the 70?

YESSICA: Right.

BRIAN: For just 7, the zero doesn't have to be there, just the 7.

LAMONT: There are two ways to make zero. This is the 7 for the tens and it [the zero] makes 70.

TEACHER: What's the other kind of zero?

LAMONT: For the ones. [*He writes 08.*]

I'm thinking, "Two kinds of zeros? Wow."

Muriel

GRADE 2, APRIL

BETH: It's like you sort of understand it, but nobody really understands it. Maybe someone will come around and figure it out. And who invented zero anyway?

I laugh and write the question on the board.

WENONA: Yeah, and who invented numbers anyway?

I write this down on the board also.

TEACHER: I need to see if I can find any information for you to read.

Several days later Lamont came with delight on his face to tell me that our librarian had seen the questions on the board and said she had a book called *Zero Is Not Nothing* (Sitomer, 1978). He eagerly went to the library to bring it back.

The next week, Henry came to me and said sincerely, "You see, Ms. Willis," holding his hands closed and then opening them palms up, "zero means there's nothing. See, there's nothing in my hands. That means zero."

I was thrilled that he had actually kept this issue in his head long enough to either talk to someone about it, or come up with that explanation on his own. It was also somehow very touching to me that he seemed to be gently offering me an explanation about something I didn't yet understand.

I guess I've written up this episode for a couple of reasons. One is simply that I love the idea that even a few of my second graders can have this kind of discussion about number. This kind of "chewing" on ideas is exactly what I most hope and work for in my mathematics (actually any subject) class. I am genuinely intrigued to have this window into some second graders' thinking about what zero is. I also am thrilled that the assertion is in the air that someone invented this zero thing, as well as the particular numbers that we use. It makes them much more accessible and "touchable."

I also wonder how making sense of zero affects children's understanding of place value. Actually, it's probably more to the point to wonder how *not* making sense of zero affects children's understanding of place value.

I am sometimes just overwhelmed with the range of ideas that bombard me in a relatively short discussion.