CHAPTER

3

What is multiplication? What is division?

As we move from the consideration of addition and subtraction to multiplication and division, the questions that were posed in chapters 1 and 2 still apply, and we extend them:

- How can a single situation be represented by different number sentences?
- When we see the different number sentences that can model a single situation, what do we learn about relationships among operations?

- What does it mean to have constructed concepts of the various operations?

Furthermore, multiplication and division come with their own questions:

- What kinds of situations are modeled by multiplication and division?
- What issues must students work through in order to make sense of these operations?
- What ideas about addition and subtraction do students bring to their work with multiplication and division?

Ponder and take notes on these questions as you read the following cases from grades 2 through 5.

CASE 12

What is multiplication?

Linda

GRADE 4, DECEMBER

As we turned away from our intense study of subtraction, I was curious to find out what my fourth graders knew about multiplication. What did they think it was? How was it different from the operations we had just been studying? These were questions I had been thinking about for myself as well as for my students, and I hadn't come to any solid conclusions yet.

From previous conversations, I had learned that multiplication, to most of these students, means a chart of facts—The Tables. Some children are happy with this definition because, since they know their facts, they "know" multiplication. Others are less comfortable with this definition because they've seen the chart but have not memorized it. I myself am unsure about the place of the multiplication facts in an understanding of multiplication. I know that even if students have mastery of the facts, that doesn't mean they necessarily understand what happens to different amounts when they are multiplied. But I'm also aware that knowing the facts makes life easier for children; besides, other teachers will expect students to know them. I decided to begin our study of multiplication by looking at the facts and trying to give them some meaning.

I flashed a large multiplication fact chart (through the 10s) on the overhead and suggested that we could eliminate the need to learn every individual fact by looking for patterns and relationships. After some discussion, the class came to the following conclusions:

Any number times 0 is 0.

Any number times 2 is that number doubled.

Everyone knows the 5s table from playing games.

We know the 10s by taking a number and putting a 0 in the ones place.

Any fact given backwards has the same answer as given forwards. ($5 \times 2 = 10$, $2 \times 5 = 10$)

Any number times 1 is that same number.

Viviana caused some debate by offering a revision of the last point, suggesting that "any number times 1 is always the bigger number." I asked her to prove that and she came to the board and wrote:

$1 \times 8 = 8$ $1 \times 7 = 7$ $1 \times 9 = 9$

It seemed true to her and most of the class was satisfied, but a couple of students looked puzzled. Kazuo wondered about 1×1 because, "What number is the bigger one?" In response, we heard "Oh" from a few students, but not much further discussion.

Then Tyrel said, "Wait—0×1 makes what Viviana said not true, because 1 is the bigger number but the answer is 0." At this point, the class decided to go with the earlier version of the last point.

As a result of these discussions, we came up with the following pared-down list of multiplication facts to memorize:

3×3	4×4	6×6	7×7	8×8	9×9
3×4	4×6	6×7	7×8	8×9	
3×6	4×7	6×8	7×9		
3×7	4×8	6×9			
3×8	4×9				
3×9					

The students were now satisfied that once they knew the answers to those facts, they would "know" multiplication. Since the period was over, I left them with that understanding, but asked them to think about whether or not there was more to multiplication than just a list of facts.

The next day I began math by asking, "If multiplication is more than a chart of facts, what is it?"

Claudia answered, "A tool for math."

Tyrel suggested, "A faster way to add."

"How is multiplication a faster way to add?" I asked the class. In the back of my mind, I'm wondering, why isn't multiplication an *action* for the students, the way adding is? Why is it a *thing*—a tool or a shortcut?

Then Kadeem spoke up. "I want to talk about what Tyrel said, about a faster way to add. What if you wanted to find out 100 plus 5 more, fast? How would you multiply?"

I thought that was a great question, but the class didn't take it on. Instead, Kandi said that she wanted to show 50 × 5. She went to the board and wrote:

$$\begin{array}{r} 50 \\ \times\ 5 \\ \hline 100 \end{array}$$

Then Kandi explained, "5 plus 5 is 10 and 0 times 5 equals 0. Add 0 on the end of 10 and you get 100."

Rashana said, "I need to show what I think. 5 times 50 is not 100." She went to the board and wrote:

$$\begin{array}{r} 50 \\ \times\ 5 \\ \hline 250 \end{array}$$

Explaining her work, Rashana said, "5 times 0 is 0. 5 times 5 is 25."

Eric disagreed, saying, "I don't understand what Rashana means, but I do understand what Kandi said. The answer is 100."

This discussion continued for the rest of the period, and in the end the class seemed to conclude that the answer to 50 × 5 is 250. However, I'm not satisfied with the arguments that were offered for one position or another. I think the children who were most confident about the procedures they memorized last year convinced the rest of the group. I am still left wondering, What *is* multiplication? What should my students understand about that operation? And how will they learn it?

C A S E 13

How do kids think about division?

Georgia

GRADES 3 AND 4, OCTOBER

In the past I have begun the year by having kids do projects, such as making maps or building pendulums, that integrate math topics with other areas of the curriculum. I did this because I wanted to see what ideas kids were thinking about before I began pursuing my own agenda. Kids were certainly invested and interested in the projects, but the specific math topics that I could expect to be covered were unpredictable. Although I love to teach math this way, and there is great potential in this approach, I decided instead to begin this year by focusing on the four operations. I shifted away from a project-based math curriculum so I could study the mathematics the children were doing. I chose to give my own creative ideas a rest to investigate the children's ideas more closely.

As I discussed my plan with a colleague, she seemed reassured by this steady, reliable way of thinking about math. Last year she clearly didn't trust that kids would get enough exposure to the four basic operations while pursuing big, in-depth projects. So she was feeling comforted—until I said I was particularly interested in learning how my eight-, nine-, and ten-year-old students thought about division. This horrified her, because she believes that children need to be taught the operations in a certain order. In her mind, first comes addition, then multiplication, then subtraction, then division.

My initial idea was just to expose kids to division problems early in their experience with numbers so that the process would be familiar when they came to the numbers that themselves implied division—decimals and fractions. But, through my classroom research, I'm learning much more about how kids think about division, what they call division, and how they define division. Much of this learning comes from my observations of their written work.

Each week I have given the students story problems that I considered to be division problems, problems I myself would solve by using division. Here are some examples of the problems and some kids' responses to them.

? Jesse has 24 shirts. If he puts eight of them in each drawer, how many drawers does he use?

Vanessa wrote: $24 - 8 = 16$, $16 - 8 = 8$, $8 - 8 = 0$, and then wrote 3 for the answer.

? If Jeremy needs to buy 36 cans of seltzer water for his family and they come in packs of six, how many packs should he buy?

This time Vanessa wrote: $6 + 6 = 12$, $12 + 12 = 24$, $24 + 6 = 30$, $30 + 6 = 36$. I still need to ask her how many packs that gives her. But what made her add this time and subtract last time?

Other students use these same methods. Is it significant that sometimes they add and sometimes they subtract? What are their choices based on? I thought the problems about Jesse's shirts and the six-packs of seltzer water were the same kind of problem, and yet students treated them differently. On reflection, I wonder if the total number (24 or 36 in this case) affects how kids approach the problem. If the total number is a familiar one, do they subtract (until they get zero), and if the number is less familiar, do they add, building up to the number, as Vanessa did for the second problem?

? Joni wants to build some bookshelves for her friends and family. If she bought 36 boards and she needs 4 boards for each bookshelf, how many bookshelves is she going to make?

Cory had clearly tried approaching this problem more than one way. His paper was filled with tally marks, which seemed to be one way he was thinking about it, while above the problem he had written $2 \div 36 = 28 \div 2 = 14$. I decided to ask him about his thinking. I assumed he meant $36 \div 2 = 28$, but I wasn't sure how he got the 28. I wanted to find out how he was really thinking about the problem instead of making assumptions.

Cory told me, "I thought it was times." Then he reread the problem aloud. "See, that's why I changed it to divided by. If it was 4 divided by, I would probably use 2 first so it would be easier. First I would do 2 divided by 36, and that equals 28, and 2 divided by 28 equals 14, so that's how I came up with 14. Because I knew divided by is half of whatever the number is, like 2 divided by 100 is 50."

I asked Cory, "Does that mean that 28 is half of 36?"

"Yes," he said, "because I know half of 30 is . . . wait a second . . . no, that isn't right. It would be 15 plus another 3, that's probably 18 and divided by that, which is 9."

Cory knew that 36 is made up of 30 and 6, and half of 30 is 15, and half of 6 is 3. Then he knew that 15 plus 3 is 18. He understands division in terms of halving, as he clearly states: "Divided by is half of whatever the number is." Halving is also implicit in his comment, "That's probably 18 and divided by that, which is 9"; he doesn't even have to say he halved it. And he knows that dividing by 4 is like halving twice. What about division that isn't based on half? Would that be division to him? What will happen when he has to divide by 3?

? You go into a pet store that sells mice. There are 48 mouse legs. How many mice are there?

Matthew organized his work beautifully. He wrote a key (*m* = mice, *l* = legs) and put his numbers in columns.

1 *m*	4 *l*
2 *m*	8 *l*
3 *m*	12 *l*
4 *m*	16 *l*
5 *m*	20 *l*
6 *m*	24 *l*
7 *m*	28 *l*
8 *m*	32 *l*
9 *m*	36 *l*
10 *m*	40 *l*
11 *m*	44 *l*
12 *m*	48 *l*

Then in a neat box he wrote, "12 $m \times 4\ l = 48\ l$." Above the box he wrote the number 12. What does this say about Matthew's understanding of division? He knows that 12 is the answer, but he feels satisfied with a multiplication number sentence in which the answer is *part* of the problem rather than the *answer* to the problem. He knows how to find the answer, but instead of the number sentence I had expected, $48 \div 4 = 12$, he wrote a multiplication number sentence.

Georgia

GRADES 3 AND 4, OCTOBER

During a conversation with classmates about a similar problem, Matthew said, “This is another division problem. It’s 63 divided by 9. What number times 9 is 63? Seven.” When I asked him to explain what there was about the problem that made it a division problem, he said, “I don’t know, but it is. But my thinking is multiplication.”

What does this say about kids’ understanding of division if they use all the operations *except* division? As I look at how kids think about division, and how I had initially hoped that the work would help kids with fractions, I wonder in what way the two topics are connected. Do kids bump up against the same ideas in both division and fraction work? What does this say about how kids think about wholes and parts?

CASE 17

Are these kids or seeds?

Melinda

GRADE 2, NOVEMBER

On Halloween I had my second graders work on this word problem:

? 6 children are sharing 45 roasted pumpkin seeds. They want to share them as evenly as possible. How many will they each get?

I have to say first that I was really amazed and impressed with the industry and confidence with which the children approached the problem. Almost everyone got right to work and seemed to be engaged. Here are some things that seemed interesting to me.

Maria, Nikita, and Su-Yin worked together. They each got 45 Multilink™ cubes and snapped them together in a row. Su-Yin's initial strategy involved taking one cube off her collection of 45 so that she would have an even number left. When I asked her why she wanted an even number, she said so she could divide it in half. I asked her why she wanted to divide it in half, and what she planned to do next. She said she didn't know. I wondered whether she had some idea that she might be able to split her two even groups into three groups each somehow. On the other hand, maybe dividing something fairly made her think of halves and even numbers, without having any specific thought about needing six groups.

I never did find out, because Nikita and Maria had a different strategy, which Su-Yin adopted. Maria started making 6 piles of cubes, first breaking off a stick of 5 cubes for each. She then gave each pile 2 more, and had 3 cubes left.

I left these three girls for a while and worked with other children. When I returned to them, Maria was working by herself, drawing her 6 piles of 7 cubes, and Nikita and Su-Yin were arguing. Nikita was saying that each child got 7 seeds, and Su-Yin was saying each got 6, and that Nikita was counting a kid as a seed. Su-Yin indicated her 6 piles of cubes and asked me, "Are these kids or seeds?" I asked her what she meant, and she said that Nikita thought that they were all seeds, but that she had made each of her groups by putting a cube to be a kid in the center and then putting seeds around it. Therefore, her groups were 6 seeds and 1 kid each.

I was interested in Su-Yin's plan for representing and solving the problem, and I wondered what she had thought about in the process. I asked her to explain and show me exactly what she had done and thought from the beginning. She said she got 45 cubes, then put the 45 cubes in a big pile and made sure there were 45. I asked her why she had to have 45, and she said, "That's how many seeds there are." She explained then that she wanted to make groups by putting a kid in the middle and putting seeds around it, so she started to take cubes from the 45, and then said, "Oh! I need to get 6 more cubes!" I asked her why, and she said, "Because those *are* seeds!"

My question is, Is this about keeping track of units again? Su-Yin had to keep track of the numbers 45 and 6 from the start of this problem, and eventually there were some 7s and a 3. She had to think not just of the numbers and how they relate to each other, but also what they represent in this problem. I think that her plan to represent a child and then give the child a fair share of cubes representing seeds was a good one.

But was it too much for her to hold onto at once? First she didn't realize that she couldn't take any of the 45 cubes to make children, because those 45 represented the whole collection of seeds. Then, in her representation, she thought each group of 7 cubes showed a child and his or her seeds, still not realizing that "children" couldn't be taken from the "seeds," and that all the cubes had to be counted as seeds in the end. It was a lot to hold onto!

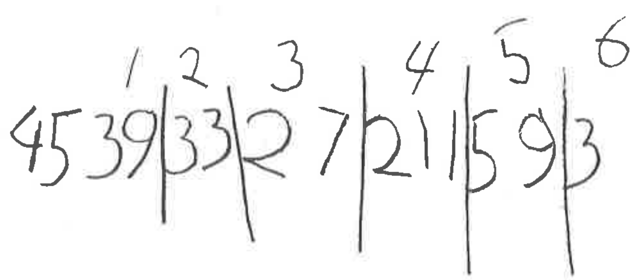
Melinda

GRADE 2, NOVEMBER

I am also intrigued by the fact that Nikita and Su-Yin were so clear about what they were disagreeing about. Su-Yin's statement that Nikita was turning a kid into a seed showed some clarity about what was going on and where the confusion was, even though it was in fact *she* who had turned a seed into a kid. I also wonder why Nikita was so sure she was right about the problem, the solution, and her representation of them, but could not convince Su-Yin.

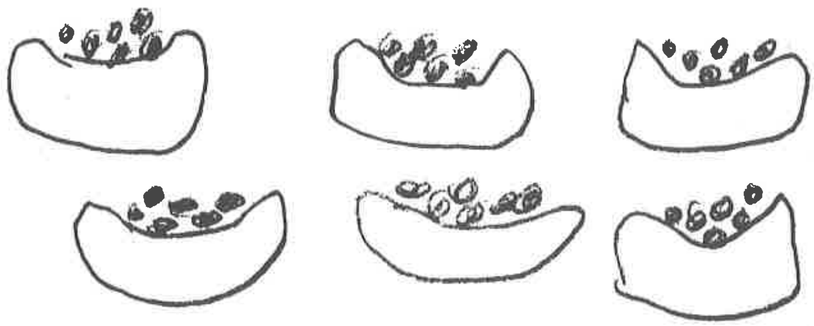
Meanwhile, Derrick and William were having an interesting time with the problem. William, who is fairly able but not very confident, said immediately, "I have an idea!" His idea, which he shared with Derrick, was to start with 45 and keep taking off 6s. They did *not* want to use manipulatives or draw pictures. They did use William's plan, and subtracted 6 repeatedly from 45 "in their heads." They ended up with a list of numbers, run together and hard to read:

William did say that he took out 6s because that was 1 seed for each child. I wasn't sure whether their list of numbers told him how many seeds each child ended up with, but my colleague Lydia was in the room, and she had a conversation with him about it. She later explained to me that he started to divide their list of numbers in a way to show that each time he subtracted 6, he was giving each child 1 seed. However, he made a mistake about where to divide the run-together numerals, and ended up with 6 groups. It didn't bother him that the numbers he created by dividing up his string did not make sense in the context of the problem.



Melinda

GRADE 2, NOVEMBER



When I joined William and Derrick, they had their list of numbers and had drawn 6 little bowls with 6 little seeds in each. I asked them what the bowls were and how many seeds they had in their picture. William replied that the bowls were the seeds for each kid. Derrick figured out how many seeds they had drawn by adding 6s, and said there were 36. William agreed. Neither boy seemed bothered by the fact that they had accounted for only 36 seeds, when there were 45 in the problem.

I asked them, "How many seeds would be left then?" They figured out that there would be 9 by counting on from 36 up to 45. When I asked what they could "do" with the extra 9 seeds, they disagreed. William suggested that each child could have one more seed, but Derrick said, "No, there's only enough for one kid. You would need another kid." I asked why, and he said because there was only "one more 6." He decided that maybe one of the existing kids could get 6 more seeds, or 12 total, and that 3 could be thrown out. For some reason, he was unwilling to divide up a group of 6 seeds in order to give each child one more.

Apparently Derrick convinced William to give one person 12 and everyone else 6 (perhaps because I joked that maybe one of the kids was bossy or bigger or hungrier). When William explained his thinking to the class (I had him put out cubes to show his bowls of seeds), he had a group of 12 and 5 groups of 6. He went back and forth between labeling his groups of 6 as one seed for each kid and as a single kid's share of seeds. I think he was confused because he initially took out groups of 6 with a plan to give one seed in the group to each kid, and to do this as many times as possible, but then began to see those original piles of 6 as the seeds *for* each kid. I wondered why he didn't distribute 6 of the remaining 9 seeds in the end. Why did he stop distributing seeds when each of the 6 kids had 6?

Melinda

GRADE 2, NOVEMBER

Again, I have questions about how children can keep the "whole" group of seeds in mind while they think of particular ways to divide it into parts. There are many issues for children to sort out: As they start dividing up the whole, what do their groups mean? Are they collections for each child? One seed for each child? Are the children themselves represented in their diagram or model of the problem? Although there were many confusions while these second graders worked on the problem, I have to say that they were remarkably able to make sense of a very complex question.