CHAPTER

2

Addition and subtraction as models

CASE 8	Red, blue, and yellow paint	*Jane, Grades 2 and 3, October*
~~CASE 9~~	~~Enough for the class~~	~~*Melinda, Grade 2, October*~~
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The children we read about in chapter 1 could solve a variety of problems through counting. In chapter 2 many of the children have developed more sophisticated understandings of operations. As we examine the arithmetic these children use to represent problems posed to them, more questions arise. One set of questions involves mathematical issues for ourselves:

- What does it mean to model a situation with an arithmetic sentence? In what ways does the arithmetic match the situation?
- Can a single situation be modeled by different number sentences?
- Can a single operation model different kinds of actions?

A second set of questions involves the children's processes of putting ideas together:

- In these samples from grades 1–3, what can we observe about the ways in which children's ideas develop?
- What appear to be some of the difficulties children face as they learn about addition and subtraction?

CASE 8

Red, blue, and yellow paint

Jane

GRADES 2 AND 3, OCTOBER

The second- and third-grade students in the multi-age class were just beginning to settle into the routine of their class. The classroom teacher and I had divided the 24 students into two equal groups: those who could add and subtract with regrouping went with their teacher, and those who couldn't stayed with me, their principal. My group consisted of four third graders and eight second graders. I decided to present a lesson on drawing pictures to solve problems.

I passed out large sheets of white paper and crayons. "I'm going to read you a problem," I said, "and I want you to draw a picture to show how you would solve it." I read this problem:

> ? The class is going to make a large mural. They get 4 containers of red paint, 3 containers of blue paint, and 1 container of yellow paint. How many containers of paint did they get in all?

"Wait, how many containers of red paint?"
"How many blue?"
"How many yellow?"

I reread the problem, and the students began to work. As I walked around the room to see what they were doing, I noticed that, in typical fashion, some of the students were very carefully drawing their containers; others were making only circles to represent the containers. All of them were connecting their pictures to the problem.

Next, I asked the students to write a number sentence that would show how to solve the problem. The students went to work. Again, I walked around the room and noticed that students were writing several different sentences. I wondered if they would see any differences among their number sentences, and if they did, how they would explain them.

When we came together as a whole group, I asked for volunteers to show their number sentences. Josh's hand went up first. I smiled. Josh worked with me last year when I was a classroom teacher, and he has made so much progress. Last year at this time, he would never have raised his hand first.

Josh wrote his number sentence on the board:

$4 + 3 + 1 = 8$

I asked him to explain why he wrote the problem that way. "Well, there were 4 red containers—that's the 4, and 3 blue containers, and 1 yellow container. So 8 paints in all."

Josh's classmates shared their number sentences: $1 + 3 + 4 = 8$, $4 + 1 + 3 = 8$, and other permutations of the three addends. The order seemed to depend on either how the students had drawn their pictures, or the order they had heard when I repeated the problem. "Do all these numbers make sense for this problem?" I asked. Lance started to shake his head no, then stopped and looked puzzled.

"It depends on which ones they got first," Charlene said, and I asked her what she meant. "They went to get the red first so it comes first, then the blue, then the yellow."

"I thought she got them all at the same time," said Jacinta.

"That doesn't matter; it's just how many they have in all," Malia said. When I questioned her further, she said, "If she gets red, blue, then yellow, she has 8, and if she gets blue, then yellow, then red, she still has 8."

"What if they mixed the paints?" Clarissa asked. The group discussed this for a while, but decided that the paints wouldn't be mixed for the mural.

Then Shiro shared his number sentence, $4 + 4 = 8$. I asked if Shiro's number sentence matched the problem. The group had different responses. Some said yes, his number sentence did match the problem because he had 8 containers in all. Others said no, because they couldn't see the various colors of paint. "It's like he mixed the blue and yellow," Clarissa said. "That's what it would look like."

"But that's not the same problem," Charlene objected. "The problem didn't say they were mixed."

Melinda

GRADE 2, OCTOBER

They showed a lovely ability and willingness to take numbers apart and put numbers together. They were clearly thoughtful about the problem and had made sense of what was being asked. But they still didn't figure out how many cubes there were in all!

I am not sure what surprises me more—that so many children don't think explicitly about the whole or the total when solving these problems, or that it never occurred to me that they didn't have to.

CASE 10

Valentine stickers

Jody

GRADES 1 AND 2, JANUARY

When the children arrive each morning, right away they start work on the problem of the day, which I have written on a chart or on the chalkboard. One day last week, I had this word problem displayed:

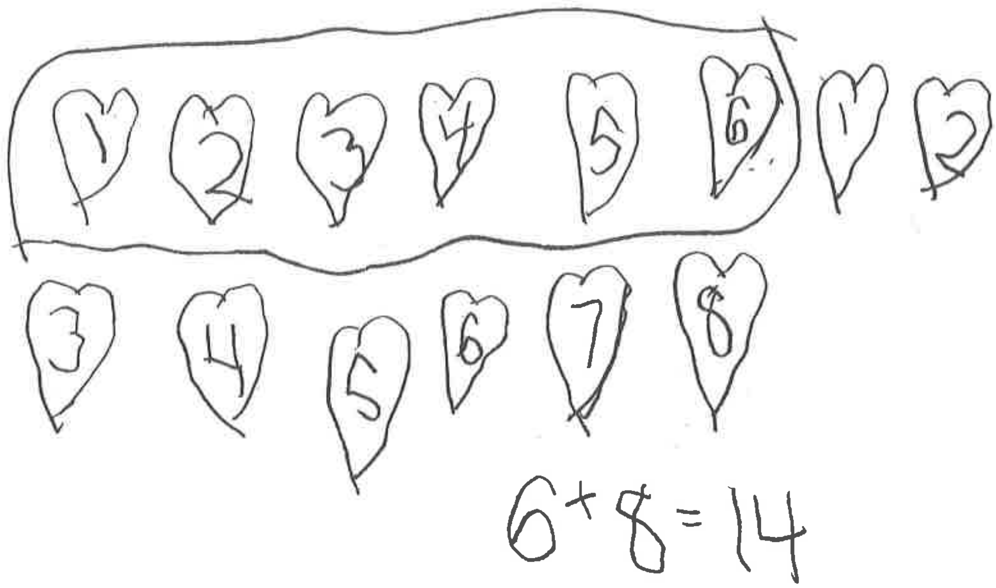
> Sabrina and Yvonne have 14 stickers when they put their stickers together. Yvonne has 6 stickers. How many stickers does Sabrina have?

Solving the problem of the day has become a routine in my class; after settling in, the children just go and get any materials they need (cubes, links, counters) to solve the problem. They know that they have to keep a record of their strategies for solving the problem, using models, pictures, words, or number sentences, so that they would be able to explain their thinking process to someone else.

This is how Latasha, a first grader, solved the problem. First she drew 14 hearts on her paper (see next page). Then she wrote numbers 1 to 6 inside the first 6 hearts. Then she started again, writing from 1 to 8 on the remaining hearts. She marked off the first 6 hearts. Below her picture of hearts, she wrote 6 + 8 = 14.

Jody

GRADES 1 AND 2, JANUARY



Most adults would think of this as a subtraction problem, but Latasha represented it with an addition sentence. When I saw what she was doing, I wanted to make sure that she was clear about her process and that she understood the problem. I asked Latasha what the 14 was. She said that this was the number of stickers Sabrina and Yvonne had together. When I asked her how many stickers Sabrina had, she quickly pointed to the hearts labeled 1 to 8 that she had not put a circle around and said, "Sabrina has 8 stickers." Her responses assured me that she understood what the problem was and that her strategy was clear to her.

Jessie, a second grader, did not draw a picture for this problem. Instead she wrote the following on her paper:

> If Yvonne has 6 stickers and they have 14 altogether, I figured it out by minusing 6 from 14.
>
> 14 – 6 = 8
>
> I also figured it out like this:
>
> 6 + ____ = 14
>
> I know that 6 + 6 = 12, add 2 more = 14, so
>
> 6 + 8 = 14

Having known Jessie for over a year now, since she was in my first-grade class last year, I know that she feels comfortable explaining her thinking process using numbers. Sometimes she uses pictures or other representations, but for this particular problem she didn't, and I felt she understood the problem. Jessie is used to explaining different ways of solving a problem. The numbers involved in the problem were low enough that she could visualize them without having to use objects or pictures. She knows her number combinations and she was able to solve the problem with both addition and subtraction. She also used what she knows about doubles to get her answer.

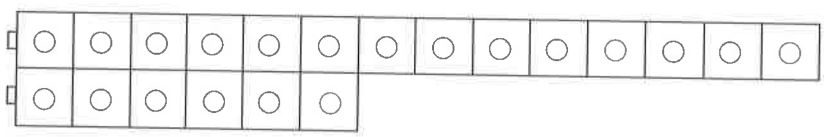
Cecile, another second grader, did not use pictures or objects either. She explained her way like this:

> I know that 7 + 7 = 14, so I took 1 from one of the 7s and put it on the other 7 so now it is 6 + 8 and it's 14.

I asked Cecile what she was thinking when she started with 7 + 7 = 14. She said, "7 + 7 is easier for me to think about and that makes 14, so if I move 1 from one of the 7s to the other, I have 6 + 8 and that is 14."

Cecile feels comfortable with renaming addends that add up to the same number. One of our daily routines is thinking of many different ways to rename a number using doubles, addition, and subtraction. Cecile often thinks of numbers that are easier and more convenient for her to deal with, to help her solve problems.

Maya is a first grader. For the sticker problem, she took 14 cubes and made a tower. Then she took 6 cubes and made another tower. She lined up the two towers next to each other, like this:



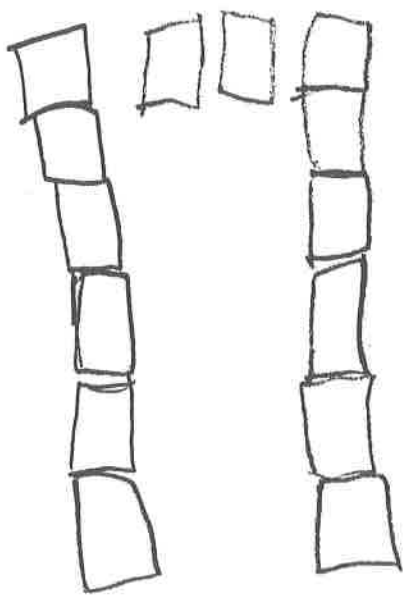
When she counted how many cubes there were beyond her 6-cube tower, she found that there were 8.

Jody

GRADES 1 AND 2, JANUARY

When I saw what Maya had done, I was struck by the fact that her model consisted of more than 14 cubes. I asked her what the 8 cubes represented, and she said it was Sabrina's stickers. She said that the 6-cube tower below the 14-cube tower was only a way for her to remember how many stickers Yvonne had. She was using the extra 6 cubes as a marker so that she could easily see how many stickers Sabrina had. I then asked her what the 14 cubes represented, and she said that these were the stickers that Yvonne and Sabrina had together. Maya could explain her process clearly by using objects, but when I asked her if she could tell me a math sentence that showed what she had just done, she was not able to do it.

Joachim, a second grader, had drawn this on his paper:



Joachim's model confused me at first, so I asked him to explain his process. This is what he said:

> I took two 6s and added them. That is 12. But this is not the correct number so I added 2 to the 12 and it is 14. So now it is 6 + 6 + 2 = 14. And 6 + 8 = 14.

Joachim's way is similar to Cecile's. They both relied on their knowledge of doubles to get to the right number.

I asked Damien, who had chosen to use a number strip, how he solved the problem. He put his finger on 6 and said, "I started counting up from 6 until I got to 14, and I counted 8." As I listened to him count, he said "1" as he pointed to 7, "2" as he pointed to 8, and kept moving up until he said "8" as he pointed to 14. Damien also made 14 circles on his paper and counted off 6 by putting a line across each of 6 circles. There were 8 circles left.

Antoinette used cubes to help her solve the problem. She explained her process like this:

> Since Yvonne has 6 stickers, I took 6 cubes. Then I said, how many more cubes should I take to get to 14, and then I counted up until I got to 14, and there were 8 more.

When I asked Antoinette to show me how she counted up from 6, she said, "6 . . . [*pause*], 7, 8, 9, 10, 11, 12, 13, 14." Her way of solving the problem is similar to Damien's first way. Although they had different materials, both used counting up to solve the problem.

Kim, a second grader, and Dylan, a first grader, were working together and role-playing the problem, pretending that one of them was Yvonne and the other was Sabrina. They counted 14 teddy bear counters together. Then Kim said, "I am Yvonne and you are Sabrina. I have 6 teddy bears. How many teddy bears do you have?" Dylan, acting as Sabrina, counted his teddy bears and he exclaimed, "I have 8. So Sabrina has 8 stickers!" They were having a good time pretending, and when I left them, I asked them to try to think of another way to solve the same problem.

All the children had appropriate solution methods. They used methods that were familiar to them; some used number combinations that were easy for them to think about. They understood the problem and were able to explain their strategies and represent the problem in different ways. My goal for all my students is that they feel comfortable in communicating their thinking process while also expanding their repertoire of strategies for problem solving. I encourage them to try solving a problem in more than one way and to share their strategies with someone else. I also would like my students to explore the properties of addition and subtraction. Jessie, who used the operations, knew that the problem could be solved by either addition or subtraction.

CASE 11

Rocks, trips, and writing groups

Dolores

GRADE 3, OCTOBER

Each new class and school year involves lots of getting to know each other and setting systems into place. The expectations for math in my classroom include being respectful of everyone's ideas (even errors), communicating about thinking, trying alternative strategies, being flexible with ideas and numbers, and enjoying the exploration of numbers during problem solving.

Every year I encounter the widely held belief that the faster you can get an answer, the smarter you are. And almost every year I encounter girls who have already come to believe that they can't do math or they don't get it. Much of my energy goes into ridding the class of those notions. Instead, students begin to feel that success is related to making mathematics make sense.

This year we have worked on several problems that caused many children to ask, "Do I add or take away?" The structure of my questions is not the typical form of providing two addends and asking for the sum, or giving a quantity and finding out what is left after some amount is taken away. Following are three of the word problems we have worked with so far.

Dolores

GRADE 3, OCTOBER

I keep many rocks on my coffee table at home. Recently friends came by and brought me more. I am sure 27 of them are new. I now have 54 rocks. How can I find out how many rocks I had before my friends came?

I took a trip to visit my dear friend who lives exactly 30 miles from my house. I used a trip counter in my car to know I stopped after precisely 15 miles to buy my friend a pumpkin. When I got back in my car I was wondering, like lots of kids always ask, are we there yet? and how much further? Can you tell how long the rest of the trip would be?

Let's say there is a special writing group that our class could join if we had 37 kids. How many more kids would we need to find? [*Our class has 24 kids, a number all my students know from daily counting routines.*]

My students have come up with many ways to solve these problems. Initially, they all "needed" me to tell them whether to write an addition problem or a subtraction one, but they got no answer from me on that question. I did encourage them to find several different ways to go about it. I said, "I'll be interested to see how you solve it." They could decide on making pictures, setting up tally marks, arranging cubes, counting on their fingers, or writing calculations.

For the rock problem, quite a few children started individual work in their math journals by making 54 tally marks to stand for the rocks. Some crossed off the first 27 marks and others crossed off the last 27. They came out with 27, or some number that was very close to 27. (Writing or counting too fast or too slow has been recognized by the class as a pitfall to avoid.) Some others had written out a string of numbers from 1 through 54, crossed off 1 through 27, and renumbered 28 as 1, 29 as 2, and so forth.

Joshua was "all done" very quickly (see his work in Figure 6). I was curious about his work because we had read, reread, and paraphrased the question as a whole class before any work began. I was more than a little surprised to hear Joshua report that a previous teacher had said, "Just check for two numbers. If the first number is bigger, take away. If the first number is littler, add." OK. That was my first reminder this year that I'll always be asking kids to use logic and reasoning, which may very well be competing with "helpful hints" or "shortcuts" taught by others.

Dolores

GRADE 3, OCTOBER

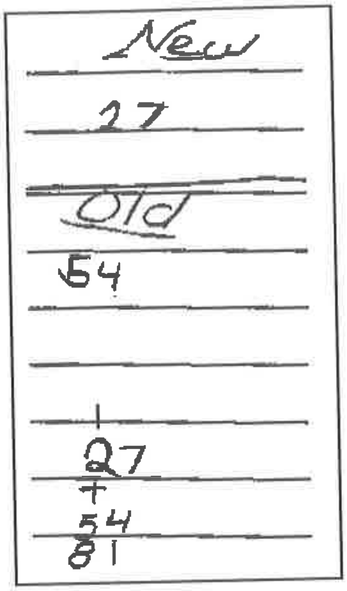


Figure 6. Joshua's work on the rock problem.

A few students, like Dimitri, drew rocks or dots to stand for rocks (see Figure 7). These students had 54 as the total in mind and put a space between 27 and the rest of the rocks needed to reach 54. They finished by counting up just the unknown amount beyond 27. There were some counting errors, which were quickly detected during group-sharing time. I was interested by the reaction of several children that 27 *couldn't* be the answer, because 27 was the number of new rocks and it couldn't be "right" for the number of old rocks, too. I wondered why. They weren't able to say; third graders need practice in articulating the reasons behind what they sense is right or not. We work on this constantly.

Several children tried to move from marks on their pages to the more abstract recording of what happened using only numbers, but they ran into difficulties; an example is Molly's work in Figure 8. The numbers on Molly's page do not reflect what her 54 Xs and tally marks show. She also lost the meaning of the 4 and 7 in 54 and 27, and simply took the smaller digit from the larger. This showed me an area where more understanding was needed. The context seemed to have been lost.

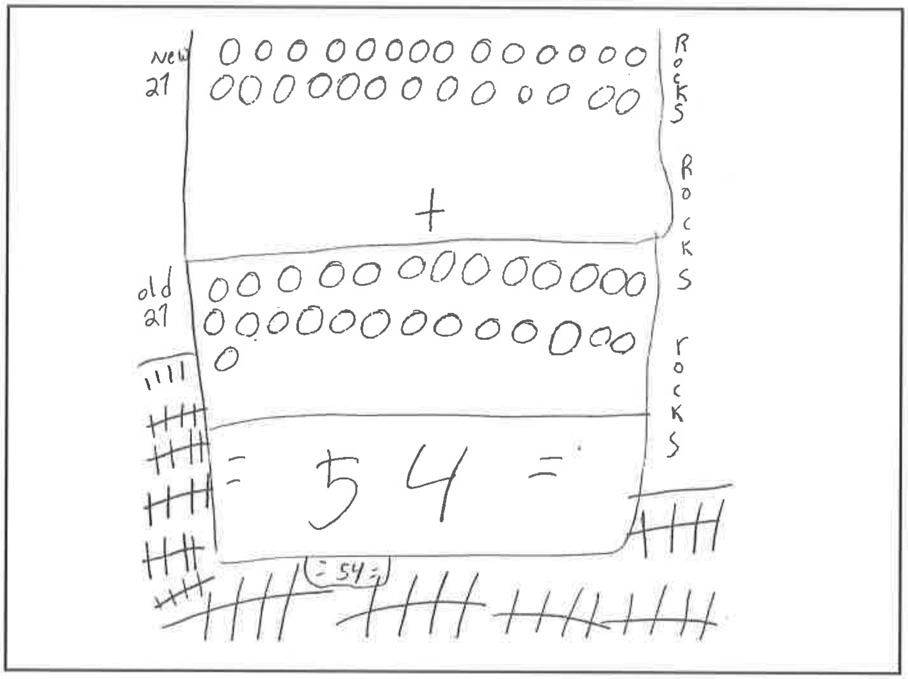


Figure 7. Dimitri's work on the rock problem.

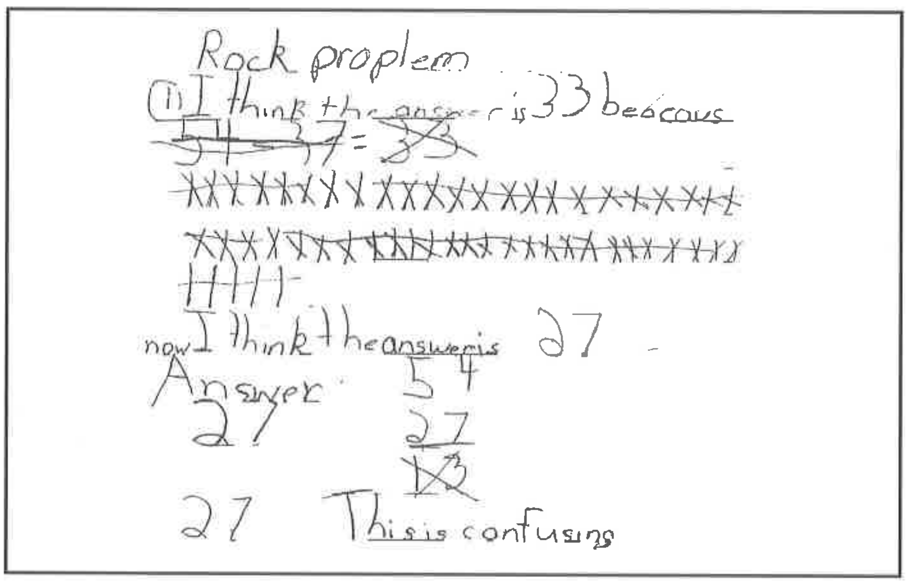


Figure 8. Molly's work on the rock problem.

Another issue that appeared in students' work on all three problems involved knowing where to start counting on. Did counting on start at the 27th or the 28th rock? Did it start at the 15th or 16th mile? Did it start on the 24th child or the 25th?

For the problem involving the second stage of a 30-mile trip, Shareena first drew all the miles for the entire trip. Then she separated off the miles driven in the first stage of the trip by drawing a line after mile 15 (see Figure 9), and started counting the remainder at mile 16.

Some of my students were not able to take the mental leap to an abstracted concept of the first part to count onto. They needed the physical objects, marks, or numbers. By contrast, Allison (Figure 10) can count on and doesn't need to have the representation of 1 to 15. There is a certain level of confidence demonstrated by students when they know the 15 miles have been driven without showing them.

Allison and her partner Christopher changed the problem by stopping for coffee instead of a pumpkin, a change that doesn't affect the outcome at all. But when Christopher (Figure 11) tried to use the same reasoning as Allison, he ran into trouble. When he drew his representation of the rest of the trip after stopping for coffee, he counted the 15th mile as another mile traveled, rather than as the *starting point* to go to mile 16.

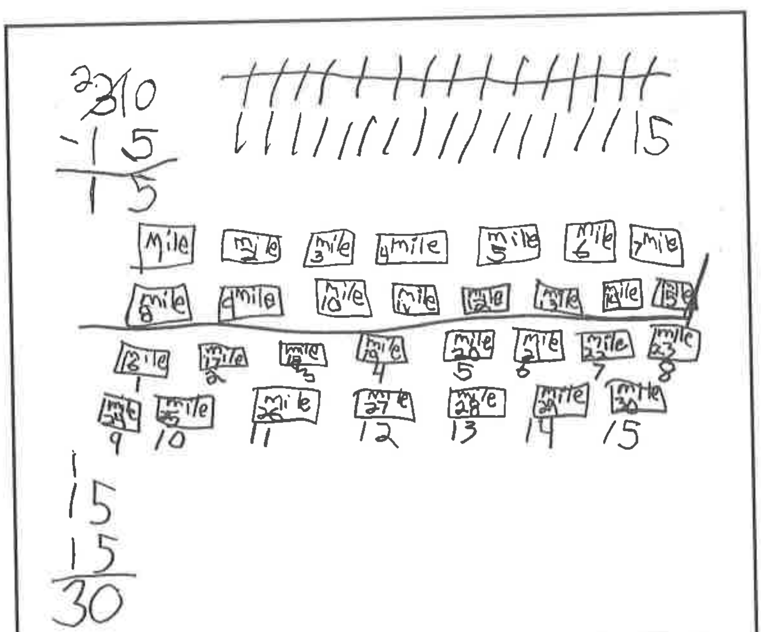


Figure 9. Shareena's work on the problem about the 30-mile trip made in two phases.

Dolores

GRADE 3, OCTOBER

This gets rather complex as students try to decide if they are counting the mile markers themselves or the intervals between. Christopher did go on to see if 15 miles and another 16 miles would get us to the friend's house. Trouble was spotted and help was offered by peers. 375

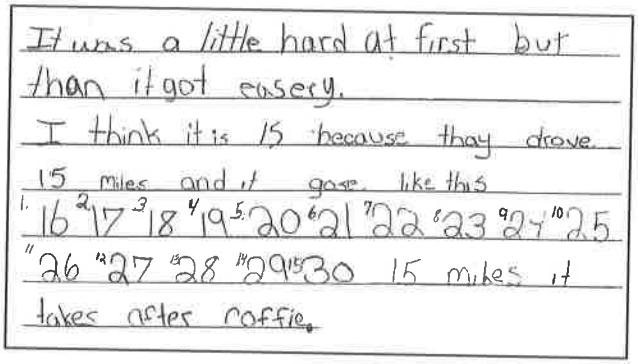


Figure 10. Allison's work on the 30-mile-trip problem.

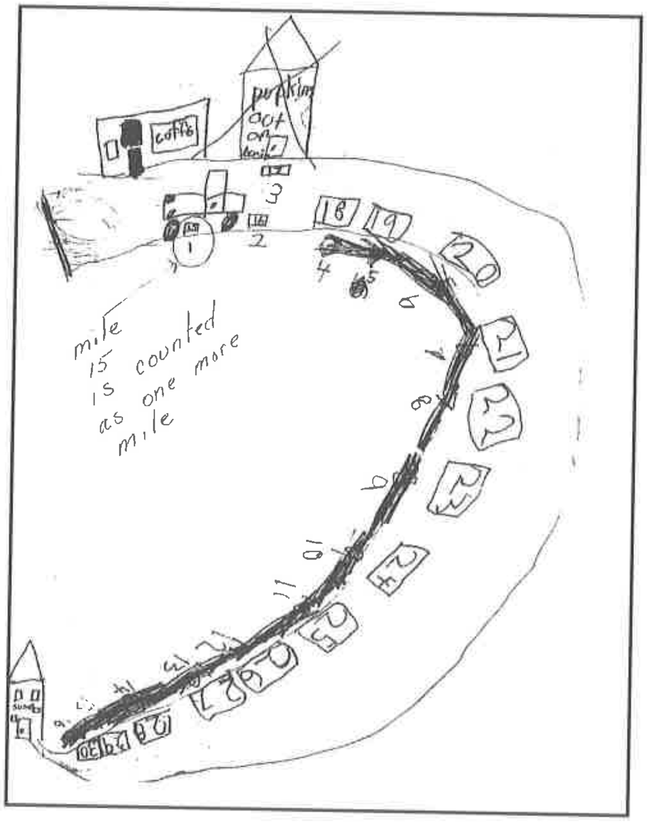


Figure 11. Christopher's work on the 30-mile-trip problem.

Dolores

GRADE 3, OCTOBER

As I got to know some of my very quiet students, I learned that a few of them pretty much believed math meant "do anything with numbers . . . I don't know why!" I frequently found children choosing any old calculation. What happened when they tried to answer some of these problems is typical of the troubles that a few third graders find themselves in each year. The calculation they wrote had no real meaning. Kalyn (Figure 12) was not visualizing a trip of 30 miles in length that had been partially completed; maybe she was partially remembering the rule Joshua had recited, about "find two numbers . . . ." But was the rule to add or take away? Kalyn's calculations don't fit the situation. I also have to wonder what subtraction really means to Kalyn if she can begin with 15, subtract 30, and still have 25. There seems to have been little, if any, expectation that this should make sense.

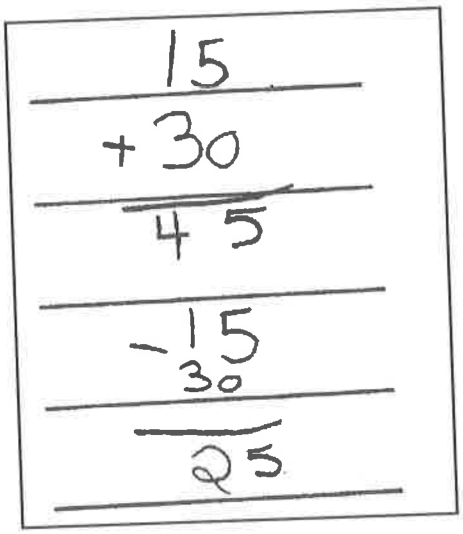


Figure 12. Kalyn's work on the 30-mile-trip problem.

By the time we got to the problem about needing 37 writers when we currently have 24, the class had spent many hours sharing solution strategies on other problems. They had shared work with partners and with the whole class. We had practiced how to ask questions if we disagreed with the work or the reasoning being shared by others. Luisa, who had previously written 27 – 54 = as the starting point for the rock problem, now solved the 37 writers problem by drawing 37 kids and circling our class of 24 (see Figure 13). Then she counted up the number of kids still needed to make 37. That was progress for this child.

Travis had needed to draw out 54 rocks for the first problem, and later needed to write out all 30 miles for the second one. Now, to solve the
third problem, Travis listed just the numbers from 24 to 37, and he knew 400
to count the 25th student as the first extra kid (see Figure 14).

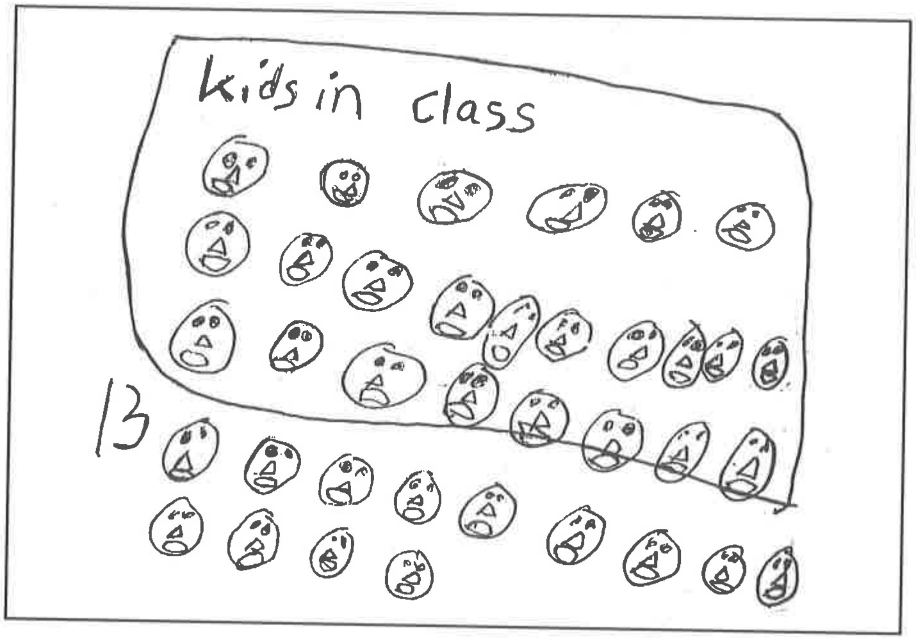


Figure 13. Luisa's work on the writing-group problem.

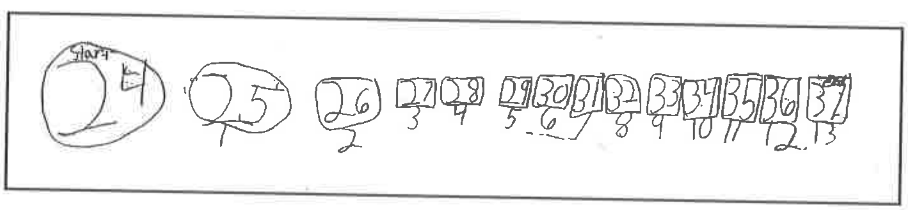


Figure 14. Travis's work on the writing-group problem.

Dolores

GRADE 3, OCTOBER

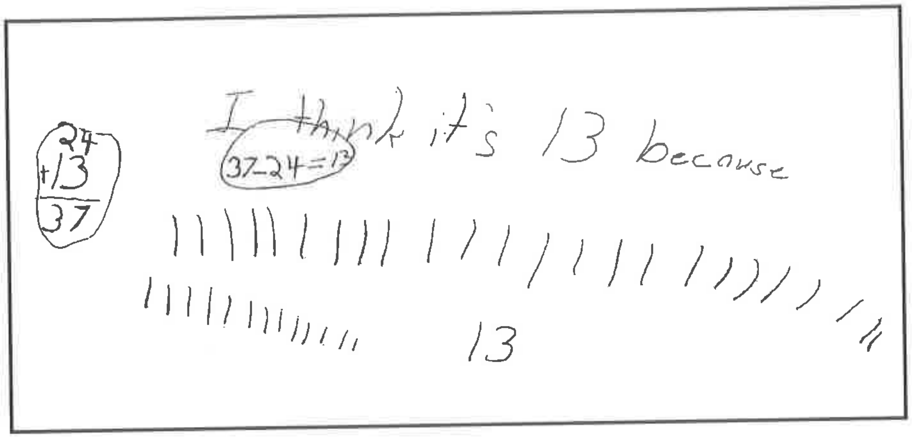


Figure 15. Doniel's work on the writing-group problem.

Several students moved back and forth between concrete representations and more abstract calculations. In Doniel's work on the third problem (Figure 15), he began to see the relationship between his tally marks and the two calculations of 24 + 13 = 37 and 37 − 24 = 13. 405

There is much work to do and many experiences to provide for this class. The first giant step has been for them to really believe that mathematics should make sense. Given that, everything else should be easier.