3

Strengthening a K–8 Mathematics Program with Discrete Mathematics

Claire Zalewski Graham

THE NCTM *Curriculum and Evaluation Standards for School Mathematics* contains a standard on discrete mathematics for grades 9–12 (NCTM 1989). But is discrete mathematics appropriate for an elementary or middle school mathematics program? Definitely, if we are truly preparing our students for the technological and information-oriented twenty-first century. During a child's elementary years the foundation is laid for strands that will be studied in depth in the future. One such strand is discrete mathematics. Discrete mathematics is not a new branch of mathematics that must be added to the existing curriculum. Rather, it is a collection of topics that most elementary teachers know something about and almost certainly already teach. These topics include counting techniques, sets, logic and reasoning, patterning (iteration and recursion), algorithms, probability, and networks. Each of these topics will be described briefly emphasizing problem solving with manipulatives. The illustrations are meant to give teachers additional ideas to implement in the classroom. It is hoped that they will be a catalyst for further exploration and development.

COUNTING TECHNIQUES

Counting techniques are used to solve a variety of problems. At the elementary school level, the emphasis should be on problems that use manipulatives or diagrams. Active involvement by the learner is essential. Using a formula to solve a problem is not appropriate for these children. After exploring numerous activities at the concrete level, most students should be ready to learn to use, or be familiar with, tools such as tree diagrams, the fundamental counting principle, and the pigeonhole principle to solve problems. Developing these strategies at a formal level is appropriate only after a great deal of concrete exploration and investigation. Two

s Program
thematics

n

Evaluation Standards for School Mathemat-
crete mathematics for grades 9–12 (NCTM
cs appropriate for an elementary or middle
efinitely, if we are truly preparing our stu-
information-oriented twenty-first century.
s the foundation is laid for strands that will
. One such strand is discrete mathematics.
new branch of mathematics that must be
n. Rather, it is a collection of topics that
something about and almost certainly al-
le counting techniques, sets, logic and rea-
d recursion), algorithms, probability, and
will be described briefly emphasizing prob-
The illustrations are meant to give teachers
the classroom. It is hoped that they will be
n and development.

NG TECHNIQUES

d to solve a variety of problems. At the
phasis should be on problems that use ma-
e involvement by the learner is essential.
blem is not appropriate for these children.
vities at the concrete level, most students
e, or be familiar with, tools such as tree
nting principle, and the pigeonhole principle
hese strategies at a formal level is appropri-
oncrete exploration and investigation. Two

examples of problems using a counting technique to determine a solution follow.

1. Using three different colored cubes, how many different ways can you arrange the cubes in a row?

   In the primary grades children can investigate this question using cubes. Through trial and error a child may demonstrate the six different arrangements. The children can use one-inch graph paper to record the different arrangements with felt-tip markers or crayons. In the intermediate grades students can use a tree diagram to help them visually account for all possibilities (fig. 3.1).

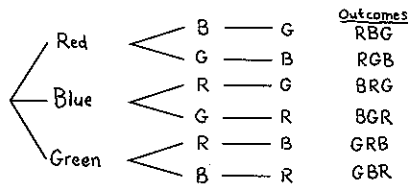


Fig. 3.1. Tree diagram

2. As a variation of this problem, plan the sandwiches for a picnic. You can use tuna fish or chicken salad on whole wheat or white bread, with any *one* of the following items: lettuce, tomatoes, onions, or pickles. How many different ways can you make the sandwiches? [*Answer*: 16 different sandwiches]

After students have had many opportunities to use concrete materials and tree diagrams to solve simple problems, a natural next step is the discovery of the fundamental counting principle, which can be used to solve similar but more complex problems. This principle states that if an event can occur in a ways and if, after it has occurred, a second event can occur in b ways, then the first event followed by the second event can occur in a times b ways. This principle can be extended to any number of events. Consider the following examples:

- If automobile license plates display six digits, what is the total number of different plates possible if only the digits 0 through 9 are used and repetitions are allowed? [*Answer*: 1 000 000]
- If automobile license plates display three letters followed by three digits, what is the total number of different plates possible if repetitions are not allowed? [*Answer*: $26 \times 25 \times 24 \times 10 \times 9 \times 8$]

The pigeonhole principle is another useful counting technique. One version of the principle can be stated, "If a set of pigeons is placed into pigeonholes and there are more pigeons than pigeonholes, then some pigeonhole must contain at least two pigeons." An example of this principle follows:

- Twelve red checkers and twelve black checkers have been placed in a bag. Without looking, how many checkers must you remove from the bag to be sure that you have five checkers of the same color?

To solve this problem, the students must consider the worst possible scenario: The first two checkers removed may be one of each color, and this may continue until four of each color have been removed. The ninth checker removed is sure to make five checkers of one of the colors.

Chapters 5, 7, and 8 contain more instructional ideas for developing counting techniques with children.

SETS

Set theory is another topic in the study of discrete mathematics. Currently, many primary-grade activities using manipulative materials explore aspects of set relationships and operations. Set concepts can be investigated throughout the elementary grades with formal vocabulary being introduced in the middle grades.

As early as kindergarten, children begin to form sets when they sort and classify objects. In simple terms, a set is a collection of objects that is so clearly described that it can be determined without question if another object belongs to that collection. Commercial materials, such as attribute blocks and Cuisenaire rods, or student- and teacher-collected materials, such as buttons, shoes, bottle caps, jar lids, pasta, keys, nuts and bolts, and spools, can be used for sorting and classifying.

Initially, objects are sorted by one attribute to form two disjoint sets. For example, a collection of buttons can be sorted by the attribute of color. Either the button is white or it is not white. The white buttons can be placed on a white mat, the nonwhite buttons can be placed on a colored mat. This illustrates two disjoint sets. Disjoint sets can also be used as an introduction to the concept of addition: If there are three white buttons and two green buttons, how many buttons are there altogether? A question like this suggests that addition can be interpreted as the combining or joining of two disjoint sets.

Next, the buttons can be sorted by two attributes—holes/no holes and white/not white. The concept of a loop or Venn diagram (using yarn or plastic hoops) can be introduced by showing that one button can be a member of both sets, thus creating a need to overlap the loops.

In the intermediate grades students can work with Venn diagrams with three intersecting circles and solve such problems as the following:

- Given the thirty-two-piece set of attribute blocks (four colors, four

› pigeons." An example of this principle

ve black checkers have been placed in a
any checkers must you remove from the
five checkers of the same color?

dents must consider the worst possible
moved may be one of each color, and this
or have been removed. The ninth checker
kers of one of the colors.

e instructional ideas for developing count-

SETS

study of discrete mathematics. Currently,
ng manipulative materials explore aspects
ions. Set concepts can be investigated
with formal vocabulary being introduced

ren begin to form sets when they sort and
a set is a collection of objects that is so
determined without question if another
Commercial materials, such as attribute
student- and teacher-collected materials,
s, jar lids, pasta, keys, nuts and bolts, and
nd classifying.
one attribute to form two disjoint sets. For
can be sorted by the attribute of color.
not white. The white buttons can be placed
ttons can be placed on a colored mat. This
int sets can also be used as an introduction
ere are three white buttons and two green
here altogether? A question like this sug-
preted as the combining or joining of two

ed by two attributes—holes/no holes and
f a loop or Venn diagram (using yarn or
ed by showing that one button can be a
ng a need to overlap the loops.
udents can work with Venn diagrams with
ve such problems as the following:

set of attribute blocks (four colors, four

shapes, two sizes), place the pieces in the Venn diagram shown in figure 3.2. Will all the pieces in the set be placed inside the circles? Why or why not? If not, list the pieces that will not be included. Can the circles be labeled so that all thirty-two pieces could be included inside the three circles?

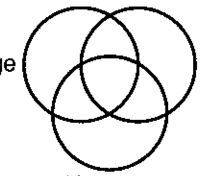


Fig. 3.2

This type of experience lays the groundwork for more involved problems to be investigated in the middle grades. For example, suppose a survey of 200 high school graduates yielded the following results:

80 students have studied French
89 have studied Spanish
64 have studied German
45 have studied French and Spanish
23 have studied French and German
35 have studied Spanish and German
3 have studied French, Spanish, and German

Questions to encourage the development of a Venn diagram might include these:

1. How many students have studied French but neither Spanish nor German? [*Answer*: 15 students]
2. How many students have studied French and German but not Spanish? [*Answer*: 20 students]
3. How many students have studied none of these three foreign languages? [*Answer*: 67 students]

LOGIC AND REASONING

The ability to reason logically is a skill necessary for daily living. Asking children to justify their thinking helps them clarify their reasoning. Useful questions are these: Why? How do you know? What makes you think about it that way? Is there another way? Questions that expect yes and no responses do not require students to explain. Low-level questions require students only to have knowledge and comprehension of the content. Encouraging students to analyze, synthesize, and evaluate the concepts will help them understand that mathematics makes sense and is not a collection of unrelated facts. Lessons in logic and reasoning can also be an integral part of every other content area in an elementary school curriculum.

Manipulative materials can be used effectively for activities requiring logic and reasoning. At the elementary school level, students can use manipulative materials and other models to help them justify their thinking. They provide concrete experiences for children to demonstrate what could otherwise be very abstract mathematics for students in the elementary grades.

All/Some/None

As young children sort and classify materials (wearing apparel, commercial objects, or "junk" collections), have them sort the objects in ways that allow discussion of the data using the words *all, some*, or *none*. Here are two examples:

1. Sort the sneakers worn by the children in the classroom. The result may lead to statements like these: All the white sneakers have laces. Some of the blue sneakers have laces. None of the black sneakers have Velcro.
2. Sort a set of cans. The results may yield these statements: All the cans have labels. Some of the cans contained food for pets. None of the cans have lids.

Constructing a class graph with information collected from the students presents data from which to draw logical conclusions. Depending on grade level, ask students either to discuss their findings or to write them first and then discuss them. Ideas for graphing include birth months, household pets, hair color, eye color, favorite subject in school, and favorite sport. The concepts of *all, some,* and *none* are appropriate in the discussions of these graphs.

If . . . Then

A concrete way to introduce a missing addend in addition problems is to use chips or other counters. Place a quantity of chips (e.g., five chips) on a mat. Ask the student to observe what is on the mat. Cover some of the chips. Ask, "If you now see three chips on the mat, then how many chips have been covered? How do you know that?" Children can work with a partner to gain additional experience with this activity.

Sequencing

Cut out from the newspaper popular comic strips that children can un

ynthesize, and evaluate the concepts will
ematics makes sense and is not a collection
gic and reasoning can also be an integral
in an elementary school curriculum.
e used effectively for activities requiring
entary school level, students can use ma-
nodels to help them justify their thinking.
es for children to demonstrate what could
hematics for students in the elementary

assify materials (wearing apparel, commer-
ns), have them sort the objects in ways that
ng the words *all, some*, or *none*. Here are

the children in the classroom. The result
e these: All the white sneakers have laces.
have laces. None of the black sneakers have

lts may yield these statements: All the cans
cans contained food for pets. None of the

ith information collected from the students
aw logical conclusions. Depending on grade
cuss their findings or to write them first and
phing include birth months, household pets,
subject in school, and favorite sport. The
e are appropriate in the discussions of these

a missing addend in addition problems is to
ace a quantity of chips (e.g., five chips) on a
ve what is on the mat. Cover some of the
ree chips on the mat, then how many chips
you know that?" Children can work with a
rience with this activity.

r popular comic strips that children can un-

derstand. Cut the frames apart and place them in an envelope with the title of the comic strip on the outside. Ask students to place the frames in a logical sequence and justify their responses. Students may want to draw their own picture sequences, cut them up, and challenge others to put them in order.

Cut apart short stories, created by either the teacher or the students, and give them to the children to put back together in the correct sequence. The sequenced story can then be read aloud to the class by the students.

Logical Deduction

With concrete materials such as attribute blocks, students can play a game called "Guess My Block." Give clues that systematically eliminate blocks until one block is found that has all the given characteristics. Depending on the grade level, blocks may or may not be sorted as each successive clue is given.

Here are two examples of sets of clues using the thirty-two-piece set of attribute blocks.

1. It is large.
   It has four sides.
   It is not a square.
   It is blue.
   It is ________. [*Answer:* Large blue diamond]
2. It is not blue or it is not small.
   It is a circle or triangle or square.
   It is not large.
   It is green or red.
   It is not a circle.
   It is not a four-sided shape.
   It is red.
   It is ________. [*Answer:* Small red triangle]

Children enjoy writing clues for other students to solve.

Inductive Thinking

Inductive thinking can begin as early as the primary grades when odd and even numbers are introduced. Manipulatives such as cubes or chips can be used to demonstrate concretely which numbers are even and which are odd. Shading graph paper can also be used as a semiconcrete model. Once a pattern has been established, students can be encouraged to extend their thinking so they can predict which large numbers will be even or odd.

Again, using the same concrete or semiconcrete materials, addition and multiplication of odd and even numbers can be explored. Once the students have reached a hypothesis, they may want to use a calculator to test it with larger numbers. Problems with more than two addends or factors can also be explored.

PATTERNING

Iteration is a technique that can be used to generate a pattern. Iteration means repeating a procedure over and over to develop a sequence.

Starting at the concrete level, attribute blocks can be used to generate a sequence. The simplest sequence is a one-difference train. Children work in groups of four for this activity. The first person selects any one of the blocks and places it on the table. The second person selects a second block that has only one attribute different from the first block and places it next to the first block. The third person selects a third block that is one attribute different from the second block and places it next to the second block. This train can be continued until either all blocks have been used or it is impossible to place another block. Additional challenges are to use all the attribute blocks in a circular train so that the last block used is also one different from the first block; to make a two-difference train using as many blocks as possible or all the blocks; to make a three-difference train with the same criteria as the two-difference train.

Numerical patterns using iteration are found in most elementary mathematics programs. Depending on the grade level, students may be asked to continue patterns such as these:

- 2, 4, 6, 8, ____, ____, ____
- 4, 7, 10, 13, ____, ____, ____
- 4, 8, 16, 32, ____, ____, ____
- 5, 5.5, 6, 6.5, ____, ____, ____
- 22, 22½, 23, 23½, 24, ____, ____, ____

Some examples for students in the intermediate or middle grades:

- Find the 50th or 100th term of a sequence.
- Given a sequence, determine what term a specific number will be; for example, given 5, 10, 15, 20, 25, . . ., what term of the sequence is 150?
- Given the first and seventh terms and assuming a constant difference, determine the first ten terms of the sequence; for example, complete this sequence: 21, ____, ____, ____, ____, ____, 63, ____, ____, ____.

ete or semiconcrete materials, addition and
ı numbers can be explored. Once the students
ey may want to use a calculator to test it with
h more than two addends or factors can also

PATTERNING

t can be used to generate a pattern. Iteration
over and over to develop a sequence.
vel, attribute blocks can be used to generate a
ence is a one-difference train. Children work
tivity. The first person selects any one of the
ble. The second person selects a second block
ifferent from the first block and places it next
erson selects a third block that is one attribute
ck and places it next to the second block. This
either all blocks have been used or it is impos-
Additional challenges are to use all the attrib-
so that the last block used is also one different
a two-difference train using as many blocks as
o make a three-difference train with the same
train.
iteration are found in most elementary mathe-
g on the grade level, students may be asked to
ese:

, ____
—, ____
—, ____
—, ____
—, ____, ____

nts in the intermediate or middle grades:

term of a sequence.
ermine what term a specific number will be; for
15, 20, 25, . . ., what term of the sequence is

venth terms and assuming a constant difference,
ı terms of the sequence; for example, complete
—, ____, ____, ____, ____, 63, ____, ____,

Recursion is another technique used to solve problems when trying to describe future results by looking at previous step(s). The Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, . . ., is a familiar example of a recursive pattern. Each new term in the sequence is determined by the sum of the previous two terms. Students can be encouraged to look for the Fibonacci sequence of numbers in many fields, including science and music. Similar patterns can be created by starting with different numbers, 3, 3, 6, 9, 15, 24, ... The building of Pascal's triangle is another activity involving recursion that has much potential for the middle grades. Many patterns can be explored, including the generation of the numbers as they emerge in the triangle, the symmetry of the triangle, and the various operations on the numbers. Researching the many applications of this array of numbers can provide an interesting investigation for students to share with their classmates. For example, how does the arrangement of the numbers relate to the number of possible outcomes in a coin-tossing experiment?

Many elementary school students have had the opportunity to learn to write Logo programs. Logo recursion is a clever way to create designs and shapes by using procedures that call a copy of themselves as a step. The procedure SQUARE, illustrated in figure 3.3, uses recursion. Students should notice the need for another statement to control the turtle and make it stop.

```
TO SQUARE
FD 100 RT 90
SQUARE
END
```

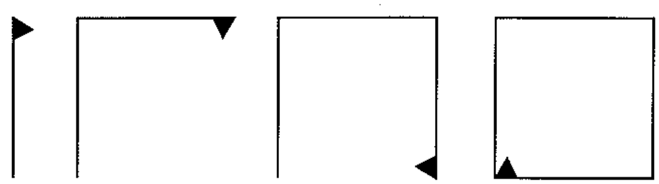


Fig. 3.3. Logo square

Further exploration can create many interesting designs from basic polygons and the use of recursion.

ALGORITHMS

The investigation of algorithms in the study of mathematics can help students organize and structure their thinking. In essence, an algorithm is a sequence of instructions that, if followed for an operation (whether mathematical or not), will always lead to a result.

At the elementary school level, children can invent algorithms to solve addition, subtraction, multiplication, and division problems. In the middle grades, students can investigate the Euclidean algorithm as a way to find the greatest common factor of a set of numbers.

In addition to these numerical algorithms, many nonnumerical algorithms can be considered and explored. Children can sequence a given set of pictures. For example, the pictures in figure 3.4 illustrate in random order the steps to follow when feeding a pet cat.

Fig. 3.4

Depending on grade level, students can be asked to illustrate a sequence, write a list of steps in a sequence, or draw a flowchart to show a sequence. Some situations to explore that require algorithmic thinking are brushing teeth, making a peanut butter and jelly sandwich, buying a birthday card for a relative, walking a route from home to a friend's home, wrapping a gift, writing a letter, dialing a telephone, and changing a flat tire.

Since a systematic approach to task accomplishment is essential in the real world, teachers are encouraged to start this process with children as early as possible.

PROBABILITY

Will you need to wear a raincoat tomorrow? What are the chances that your favorite baseball player will hit a home run this week? Probability has many applications in business, sports, and the sciences. Stockbrokers, airline staff, and weather forecasters make predictions in their work. The study of random happenings can start in the elementary grades, since many investigations and experiments can be carried out with manipulative materials. Children enjoy making predictions and having experiences with elements of chance.

ed. Children can sequence a given set of ures in figure 3.4 illustrate in random order g a pet cat.

Fig. 3.4

udents can be asked to illustrate a sequence, ice, or draw a flowchart to show a sequence. at require algorithmic thinking are brushing and jelly sandwich, buying a birthday card from home to a friend's home, wrapping a telephone, and changing a flat tire.

n to task accomplishment is essential in the uraged to start this process with children as

ROBABILITY

ncoat tomorrow? What are the chances that vill hit a home run this week? Probability has , sports, and the sciences. Stockbrokers, air- ters make predictions in their work. The study art in the elementary grades, since many in- can be carried out with manipulative mate- predictions and having experiences with ele-

At the early stages, children can experiment with materials to determine *experimental* or *empirical* probability. By actually conducting an experiment several times, children can determine the number of ways an event occurred compared to the total number of possible events. Children in middle school can still conduct probability experiments but can also begin to discuss the *theoretical* probability of an event.

Some experiments or activities to use as a basis to compute probabilities follow:

- *Dice tossing*
  Have children work as partners and toss pairs of dice 25 times. One student rolls the dice and the other records the sum of the dice for each roll on a tally sheet. A bar graph is drawn to show the results. Use discussion questions like these: What sum appeared the most? The least? Can you predict the results for 100 tosses? 300 tosses? All the partnerships then pool their data. How accurate were the predictions? Can you explain why this happens? (*Hint:* Look at the number of ways each sum can be obtained when tossing two dice.)
- *Weather predictions*
  Have students watch the weather report on television or read it from a daily newspaper to record the probability of precipitation (rain or snow) predicted each day. Discuss what probability means and observe what happens on days when the predicted probability is very high or very low. The class may want to chart the predictions and actual outcomes. After an extended period of weather watching, discuss the accuracy of weather predictions.
- *Coins and spinners*
  Carry out similar activities and experiments with coins and homemade or commercial spinners.
- *Playing cards*
  A variety of probability problems using playing cards can be acted out and discussed with intermediate or middle grade children. For example, place a standard deck of playing cards (no jokers) facedown on a table. One card is drawn. What is the probability that it will be a picture card?

NETWORKS

Networks or graphs—that is, figures that consist of points called vertices, which are connected by edges—provide excellent discovery lessons for students in the elementary or middle grades. Activities at this level should be kept informal so that the emphasis can be on developing the students' abilities to represent, explore, and solve problems using graphs. Analyzing explorations and verbalizing their thoughts will be useful in evaluating their reasoning. Later, students will explore applying graphs to a variety of disciplines and careers.

Transportation networks are a rich source of situations that lend themselves to graphical analysis. A road inspector is responsible for checking all roads within a town. The inspector needs to travel each road but would like to do it in the most efficient way so that she does not have to travel any road twice during her inspection tour. The inspector may pass through any intersection of roads as many times as necessary. Is it possible for a road inspector to do her job for the street networks in figure 3.5? Remember—no road may be traveled more than once. [*Answer:* All street networks in figure 3.5 can be traveled except *b, d,* and *f*.]

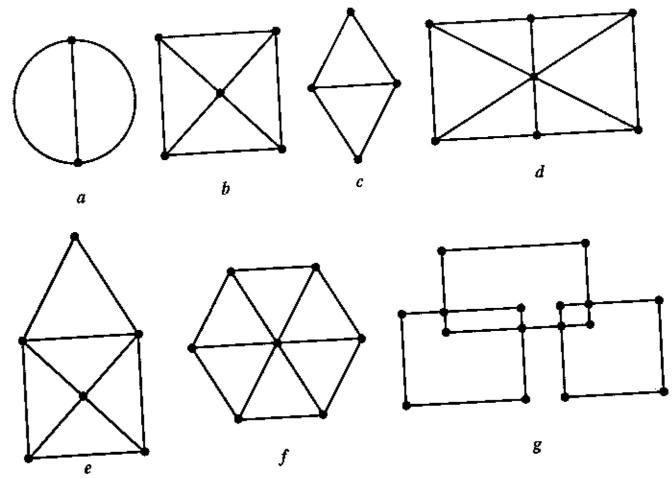


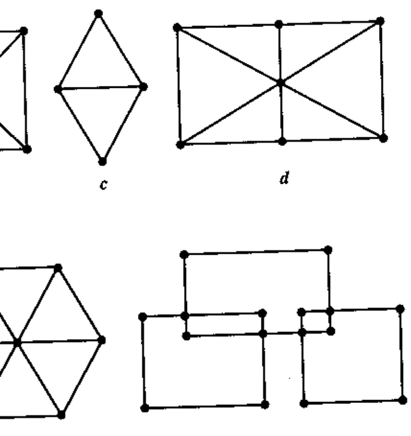
Fig. 3.5. Street networks

Organizing the data by recording them in a table may help students conjecture whether or not a network is traceable. Students may enjoy designing their own networks and having others decide if they are traceable or not.

A game called "sprouts," invented at Cambridge University in England in 1967 by John Conway and Michael Paterson, is based on graphs. The game is for two players and is started with a certain number of dots (two, three, or more). A move consists of connecting two vertices with an edge and placing a new dot anywhere on the new edge just drawn. There are two basic rules: (1) Each vertex can be the endpoint for no more than three edges, and (2) An edge cannot cross another edge. The last player able to draw an edge is the winner. Continue the game started in figure 3.6. How many more moves are possible?

nd solve problems using graphs. Analyzing ir thoughts will be useful in evaluating their explore applying graphs to a variety of dis-

a rich source of situations that lend them- oad inspector is responsible for checking all ctor needs to travel each road but would like so that she does not have to travel any road r. The inspector may pass through any inter- s necessary. Is it possible for a road inspector tworks in figure 3.5? Remember—no road e. [*Answer:* All street networks in figure 3.5 d *f*.]



g. 3.5. Street networks

ecording them in a table may help students network is traceable. Students may enjoy de- nd having others decide if they are traceable

invented at Cambridge University in England d Michael Paterson, is based on graphs. The is started with a certain number of dots (two, nsists of connecting two vertices with an edge ere on the new edge just drawn. There are two can be the endpoint for no more than three not cross another edge. The last player able to Continue the game started in figure 3.6. How ble?

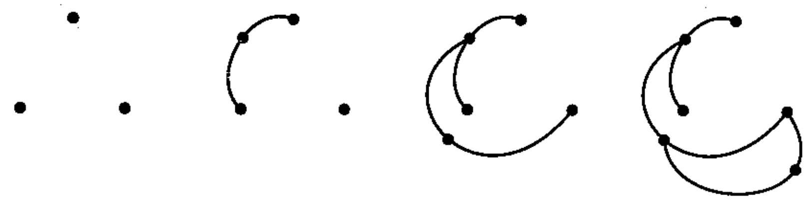


Fig. 3.6. Sprouts game

Since it can be observed that a graph divides a plane into regions, a natural follow-up to the study of graphs is the exploration of map coloring. The street networks in figure 3.5 can be the beginning of the exploration. The basic rule is to find a way to color the regions so that regions that share a boundary (a point is not considered a boundary) are not the same color. The goal is to shade each map with the fewest number of colors. How many colors are needed to color a map of the continental United States?

Networks offer interesting and challenging investigations. The experiences are rich and furnish opportunities for exploring, making conjectures, summarizing, and analyzing data. Additional examples are in chapter 4.

SUMMARY

Many concepts, problems, activities, explorations, and experiments from discrete mathematics are already present in elementary and middle school mathematics programs. What is needed is more focused attention on these topics. The examples cited here are a mere sample of the myriad of ideas that are in, or can be incorporated into, the mathematics curriculum. In time, additional discrete mathematics that is appropriate for elementary and middle school students can be added to the program.

The study of mathematics should be a broad and challenging experience for our students so that they will be prepared for the next century. Teachers are encouraged to introduce their students to the exploration and investigation of discrete mathematics topics to broaden their perspective of the study and application of mathematics.

REFERENCE

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: The Council, 1989.