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Permutations and Combinations: A Problem-solving Approach for Middle School Students

Linda J. DeGuire

"PERMUTATIONS? . . . Combinations? . . . Aaagh!" Many people have unpleasant memories of these topics. This chapter presents a problem-solving development of these topics that makes them enjoyable, understandable, and meaningful for both middle and high school students. The amount of time needed to present the material in an actual classroom will vary with the level of the students. Middle school students will need several class periods, whereas high school students may successfully assimilate the material in one class period. Prototypical solutions, teaching suggestions, and comments are interspersed throughout. In the problem solutions, the teaching suggestions appear in italics to distinguish them from the solution. The reader is strongly encouraged to read with pencil in hand—active participation through solving the problems is recommended in the exploration and development of the topics.

FUNDAMENTAL COUNTING PRINCIPLE

The fundamental counting principle asserts that if one task can be performed in m ways and a second task can be performed in n ways, then the number of ways of performing the two tasks is mn. This principle can be extended to any number of tasks. Most middle school students—and even many secondary school students—would not find such a formal statement meaningful or useful. However, if the principle is developed in relevant and concrete situations, many students will relate to it, as in the following example:

Problem 1. Barbara has 3 blouses and 2 skirts, all of which coordinate to make outfits. How many different outfits can Barbara make?

Solution. Call the blouses B1, B2, and B3, and call the skirts S1 and S2. *(With middle school students, use the words. Letters are used here to abbreviate the presentation.)* Then students can use the problem-solving strategy of listing all possibilities to arrive at the following list of outfits:

B1-S1	B2-S1	B3-S1
B1-S2	B2-S2	B3-S2

The list shows 3 groups of 2 outfits each. *(Regardless of the order the students use to suggest the combinations, write them on the chalkboard or overhead projector in an arrangement, as above, that suggests 3 groups of 2.)* Thus, Barbara can make 6 different outfits.

Alternative solution. Some students may find a graphic representation of the list more meaningful. In figure 8.1, each branch of the tree diagram represents one outfit. As with the list above, there are 3 sets of 2 branches each.

Fig. 8.1

Students need more examples of these types of problem situations, including those that extend the list or tree diagram to include three choices, as in this example:

Problem 2. A nearby restaurant offers five choices on its dinner menu—meatloaf, hamburger steak, chicken breast, pork chops, and shrimp. With each entree, the customer may choose one from each of the following two groups: french fries, rice, or baked potato; ice cream or Jell-O. How many different meals could be ordered at this restaurant?

Solution. Represent the entrees by M for meatloaf, H for hamburger steak, C for chicken breast, P for pork chops, and S for shrimp. Let the side dishes be F for french fries, R for rice, and B for baked potato and the desserts be I for ice cream and J for Jell-O. *(Give students an opportunity individually, or in groups, to list the different meals. Encourage them to first list partial meals of just entrees and side dishes. Point out orderly ways of making such lists. After the students have some of the combinations, suggest that they put dessert choices with each of the partial meals to make full meals. Finally, use the problem-solving strategy of listing all possibilities by pooling lists on the chalkboard or overhead projector. Arrange the meal combinations similar to the arrangement below to show 5 groups of 3 initially and then 15 groups of 2. The tree diagram in figure 8.2 shows the 5 groups of 3 and the 15 groups of 2 very clearly.)*

MFI	HFI	CFI	PFI	SFI
MFJ	HFJ	CFJ	PFJ	SFJ
MRI	HRI	CRI	PRI	SRI
MRJ	HRJ	CRJ	PRJ	SRJ
MBI	HBI	CBI	PBI	SBI
MBJ	HBJ	CBJ	PBJ	SBJ

There are 15 entree/side dish partial meals. Each of these 15 possibilities can be put with each of the 2 dessert choices, making 15 groups of 2. Therefore, 30 different meals are available at the restaurant.

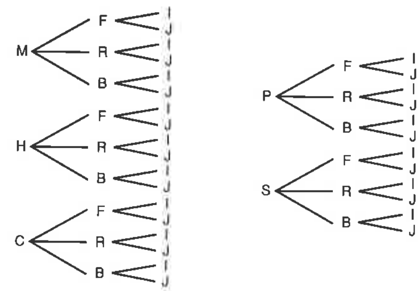

Fig. 8.2

Before more complex problems are introduced, it is important for the students to abstract the commonalities among these situations and find a more efficient procedure than listing possibilities or making tree diagrams. Many students discover the multiplicative nature of the process themselves. They readily see that the solution is reached by multiplying the number of choices at each position, or *slot*, together. Thus, in problem 1, 3 choices times 2 choices gives 6 outfits, and in problem 2, 5 choices times 3 choices times 2 choices gives 30 meals.

The fact that the product of the numbers of the choices for each slot yields the number of possible combinations is basic to the rest of the activities presented here. That is why it is called the *fundamental* counting principle. It is *very* important for the student to solve a variety of such problems by making lists or tree diagrams and counting the combinations. These solutions may be tedious, but they convince the student of the reasonableness of multiplying to find the number of combinations. When the students are sure that the results are true and do not need to continue verifying them with lists, problems like the following in which such a list would be formidable can be introduced:

Problem 3. A national organization wants to issue each of its members an ID code. The officers decide to use a four-character code that begins with a letter (not including O, in order to avoid confusion with the digit zero) and continues with three digits. They have 32 000 members in the organization. Will they be able to assign each member a different ID code?

Solution. There are 4 slots to fill here. The first slot (the letter) has 25 choices and each of the other slots has 10 choices (i.e., the digits 0 through 9). So the number of possible ID codes is 25 × 10 × 10 × 10, or 25 000. Since the organization has 32 000 members, they will not have enough distinct ID codes for each member. They need to redesign their plan and add another letter or another digit or change one of the digits to a letter. *(The question of whether the number of possibilities is enough to fit the condition or is more than is needed affords the opportunity to bring the problem solving closer to real life. Ask the students to modify the proposed plan—for example, the ID code—to be more realistic and to evaluate the proposed modifications. For example, in this problem the proposals of adding another letter or another digit will generate 625 000 and 250 000 different ID codes, respectively, each with five characters. The third proposal (changing one of the digits to a letter) will generate 62 500 ID codes, each with four characters. Unless the organization plans to double its membership in the near future, the last modification would require less computer memory space and less typing for each membership code. Thus, this code would be more cost-efficient than the other two proposals. However, if such an increase in membership is anticipated, reassigning and retyping codes and reprogramming computers for the extra character at a later time would be time-consuming and costly. In this situation, one of the other proposals should be considered. A variety of other parameters could be considered in the discussion.)*

PERMUTATIONS WITHOUT REPETITIONS

After students have learned and practiced the fundamental counting principle, they can tackle problem situations that involve permutations. In permutation situations, the order of the slots matters, as shown in problem 4. Initially, the repetition of elements should be avoided.

Problem 4. A class is having an election for class president. Four names are to be listed on the ballot—Jim, Karen, Linda, and Michael. How many ways can the names be listed?

Solution. Use the problem-solving strategy of listing all possibilities. *(Encourage students to make their lists in an orderly way. The order itself is not particularly important, as long as it gives all possibilities and makes sense to the student. Students usually need some practice with such listings, especially with small sets, such as the possible orderings of letters in the word* cat. *The practice is most useful after an introduction to permutations.)* Use J for Jim, K for Karen, L for Linda, and M for Michael. Make a list in four groups, one for each name. Within the group beginning with J, the other three names could be arranged in any of the following six ways: KLM, KML, LKM, LMK, MKL, MLK. The complete list of possible arrangements looks like figure 8.3. There are 24 possible ways (4 groups of 6 each) to arrange the four names on the ballot.

Fig. 8.3

Alternative solution. *(In many classes, the following solution may have already been proposed by some students. At this early stage of development, solving the problem in more than one way is essential. The less sophisticated solution procedure of listing all possibilities serves as a proof to convince the student that the more sophisticated procedure of multiplying the number of choices in each slot is believable and acceptable.)* First, use the problem-solving strategy of trying a simpler problem first. How many ways could the first two slots be filled? The first slot has four possibilities. The second slot has only three choices left. So, the first two slots could be filled in 4 × 3, or 12, different ways. How many ways could the 12 beginnings be finished? There are 2 names left. The third slot has two choices, and the fourth one has one choice. Thus, 2 × 1 ways exist to complete each of the 12 beginnings. Finally, putting these pieces together, we find that the names could be listed on the ballot in 4 × 3 × 2 × 1, or 24, ways.

The concept of the factorial of a number (e.g., $4! = 4 \times 3 \times 2 \times 1$) is frequently introduced as a shortcut for a product of a decreasing sequence of factors.

The introduction to permutations with a problem like problem 4 can be followed with a series of exercises to practice the newly learned procedure. The exercises can vary in situation and difficulty appropriate to the level of the students. Difficulty can be varied at least two ways—by increasing the size of the numbers involved (problem 5) and by including special considerations for certain slots (problem 6). *(Be careful about increasing the size of the numbers. The number of possibilities increases rapidly.)*

Problem 5. *(Increase the size of the numbers to the maximum possible on an 8-digit calculator without going into scientific notation or showing an* *error.)* How many ways can the letters of the word *workmanship* be arranged?

Solution. There are 11 slots to be filled, with 11 choices for the first slot and one less choice for each successive slot. Thus, 11 × 10 × 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1, or 11!, arrangements can be used, that is, 39 916 800 arrangements—a few more than it is reasonable to list.

Problem 6. *(Put extra conditions on one or more of the slots.)* How many different ways can the letters of the word *purchase* be arranged if each arrangement must begin with a consonant and end with a vowel?

Solution. There are 8 slots to be filled. Use the problem-solving strategy of breaking the problem into pieces—the first slot, the next 6 slots, and the last slot. The first slot has five choices, the consonants *p, r, c, h,* and *s*. The last slot has three choices, the vowels *u, a,* and *e*. For the middle 6 slots, 6 of the 8 letters are left from which to choose; the 6 letters can be arranged in 6 × 5 × 4 × 3 × 2 × 1, or 6!, or 720, ways. Thus, the letters of *purchase* can be arranged in 5 × 6! × 3, or 10 800, ways, each beginning with a consonant and ending with a vowel.

COMBINATIONS

Though the order of elements is sometimes a consideration, it is irrelevant in many problems. For example, in Michigan's statewide lottery, the ticket purchaser tries to choose six numbers from the whole numbers 1–45 that will match the six numbers picked on Saturday evening. On Saturday, 45 numbered balls blow around in an air chamber until first one ball—then another—then another, and so on, falls down a chute. The chute is closed after six balls have fallen. Since only the six numbers are important and not their order, this is an example of a combination of 45 things taken 6 at a time. This example is too complex for most students to use successfully as an initial attempt to solve such problems. The technique of solving a simpler problem first should be useful here.

Problem 7. Suppose a special lottery requires choosing 2 letters to try to match the 2 letters that will be picked from 4 lettered balls A, B, C, and D. How many ways could a ticket purchaser choose the 2 letters?

Solution. There are 2 slots to be filled, with 4 choices for the first slot and 3 choices for the second slot. Using the fundamental counting principle, we see that there are 12 ways to choose the 2 letters. But wait. Does it matter whether A is chosen first and then B or B is chosen first and then A? No, it doesn't matter. This list of 12 ways has duplicates in it that differ only in the order in which the letters are arranged. List the 12 ways and cross out duplicates. Partitioning a set into subgroups of 2 suggests division.

The number of combinations of 4 letters taken 2 at a time is $(4 \times 3) \div 2$, or 6.

It is unlikely that students will be able to derive the method from this one example. They need further examples before being challenged to make a generalization.

Problem 8. Suppose a special lottery requires the ticket purchaser to choose 3 letters to try to match the 3 letters to be picked from 4 lettered balls A, B, C, and D. How many ways could the ticket purchaser choose the 3 letters?

Solution. There are 3 slots to be filled, with 4 choices for the first, 3 for the second, and 2 for the third. Thus, there are $4 \times 3 \times 2$, or 24, ways to choose the 3 letters. However, some of these ways are the same except for order. As in the solution to problem 5, list the 24 ways, and group together the ways that are the same except for order. (See fig. 8.4.) Each of the 24 ways falls into a subgroup of 6 that are alike except for order. Since partitioning a set into subgroups of 6 suggests division, the number of combinations of 4 things taken 3 at a time is $(4 \times 3 \times 2) \div 6$, or 4.

Fig. 8.4. Listing the 24 ways to arrange the 3 letters

Now challenge the students to see how they could have predicted the 6 subgroups in problem 8, and the 2 subgroups in problem 7. Being able to find this number without actually listing combinations that are the same except for order would shorten the solution process considerably. At this stage many students will recognize that the number needed for the division is merely the number of ways to rearrange the 2 or 3 things chosen. Therefore, in problem 7, the 2 letters can be rearranged in 2×1, or $2!$, ways. In problem 8, the 3 letters can be rearranged in $3 \times 2 \times 1$, or $3!$, ways.

Students need to practice this new procedure with exercises that have sets small enough to allow the new procedure to be verified with actual lists of combinations. As students become confident in understanding this solution procedure, problems can be introduced in which the numbers are too large to permit listing the combinations, or in which extra conditions are placed on certain positions.

Finally, students will be able to return to the story about the Michigan lottery that begins this section.

Problem 9. In the Michigan lottery, the ticket purchaser chooses 6 whole numbers, trying to match the 6 numbers to be picked from the 45 balls numbered 1 through 45. How many combinations of 6 numbers can be chosen?

Solution. There are 6 slots to be filled, the first with 45 choices, the second with 44 choices, and so on. Before removing those that are the same except for order, we see that $45 \times 44 \times 43 \times 42 \times 41 \times 40$, or 5 864 443 200, arrangements of the 6 numbers are possible. There are 6!, or 720, ways to arrange 6 things. Each of the original combinations fits into a subgroup of 720 combinations that are the same except for order. Dividing the original number of arrangements by 720 produces 8 145 060 possible combinations of the 6 numbers. *(Ask students whether at $1 a ticket this lottery is a good bet when the jackpot begins at $1.5 million.)*

CONCLUSION

The development of permutations and combinations in this chapter is an example of teaching *through* problem solving, that is, using problem situations and problem-solving strategies to develop specific mathematical content. The solution procedures become the standard procedures students learn for future use. It is hoped that these explorations have been enjoyable and that readers will be willing to try some of these problems with their students.

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