DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume 36, 1997

Mathematical Modeling and Discrete Mathematics

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1. What is Mathematical Modeling?

When people talk about the connection of mathematics with the rest of the world, they use a number of phrases such as "applied mathematics", "problem solving", "word problems", and "mathematical modeling", to name just a few. In order to define these more precisely, and to differentiate among them, I should like to begin by describing the series of activities which seem to take place when we try to use mathematics to examine something in the rest of the world. Some situations involving discrete mathematics to which this analysis applies will be given later.

(1) The process begins with something outside of mathematics which you would like to know or to do or to understand.
   - The result is a question in the real world, well-defined enough that you can recognize when you have made progress on it.

(2) You next select some important objects in this situation outside of mathematics, and relationships among them.
   - The result is the identification of some key concepts in the situation you want to study.

(3) You decide what to keep and what to ignore in your knowledge of the objects and their interrelationships.
   - The result is an idealized version of the question.

(4) You translate the idealized version of the question into mathematical terms.
   - The result is a mathematical version of the idealized question.

(5) You identify the field of mathematics you think you're in.
   - You bring into the forefront of your consciousness your instincts and knowledge about this field.

(6) You do mathematics.
   - The result is solutions, theorems, special cases, algorithms, estimates, open problems.

1991 *Mathematics Subject Classification.* Primary 00A71, 00A05, 00A35.

(7) You now translate back into the setting of the original problem.
- You now have a theory of the idealized version of the question which you found in (3) above.

(8) You confront reality in the form of the original situation as represented by (1). Do you believe what is being said in (7)? In other words, do your results, when translated back to the original situation, fit the real world?
- If yes, you have succeeded. You tell your friends, write it up, publish some papers, get a raise, get promoted, or whatever.
- If no, go back to the beginning. Did you pick the right objects and relationships among them? Do your choices of what to keep and what to ignore need to be revisited? The way in which your theory of the idealized problem fails to satisfy you should provide some hints of where there are difficulties.

An example in which this process can be followed in detail would take us too far afield in the present context. The author's forthcoming paper [7] contains a detailed history of such a problem, the modeling steps, and the repeated modeling cycle.

2. Applied Mathematics, Word Problems, and Modeling

What have I just given is a brief outline of "mathematical modeling". When I use that term henceforth, this is what I mean. Now what is "applied mathematics"? The way the term is usually used, it begins with some idealized version of reality, translates it into mathematics, does a lot of mathematics, and, at its best, translates back; in other words, (4)–(7). Courses in "methods of applied mathematics" concentrate on the mathematical methods that tend to come up in (6) when you start with questions in physics.

What is a "word problem"? Typically, a word problem begins with a few words from outside mathematics to provide a semblance of (4), occurs in the textbook in a place where (5) is obvious, and concentrates on (6).

There is very little agreement on the meaning of "problem solving". It can be taken to mean doing a word problem, or applied mathematics, or modeling from beginning to end. Sometimes, "problem solving" refers to the process of solving mathematical problems with no reference to an external situation at all. When problem solving refers to word problems or to applied mathematics, i.e., beginning with (4), the earlier stages (1)–(3) are sometimes referred to as "problem finding", or "problem formulation".

Word problems have a history of being unrealistic, and the persistence of particular types lends itself to easy caricature. We have pipes of various capacities which can fill and empty bathtubs and John's age when Susie was twice as old as Sally. We have learned to say "center of mass", "moment of inertia" and "pendulum" with a straight face, but we go directly to the formulas without any thought in between. We make no attempt to see if our answers make any sense in the original situation because we had no original situation to begin with! That's very typical of many word problems, I'm afraid. On the other hand, what modeling requires is understanding of the original situation, an argument that the idealization makes sense, and the check that the results of the mathematical work carry meaning outside of mathematics.

3. Discrete Mathematics and the Teaching of Modeling

I believe that relating mathematics to the rest of the world is an essential part of mathematics education. We have not done our job if this aspect is not included. We ought to have word problems, traditional applied mathematics, and mathematical modeling—all three. Why? If modeling is what actually happens when you apply mathematics in the real world, why don't you just teach that? There are three main difficulties that I will discuss: mathematical modeling takes a lot of time, it requires a lot of knowledge on the teacher's part, and there is a lack of certainty in the results which, in the eyes of the public, is quite uncharacteristic of mathematics.

It is without doubt true that modeling is time consuming. So let us agree that not every problem with an applied flavor will go through the full (1)–(8) above. But in terms of a typical word problem, how do you tell a good problem from a bad one?

My answer depends on whether the problem *could* be the middle portion of a genuine model. What do I mean? Here is a sample word problem: "An electric fan is advertised as moving 3375 cubic feet of air per minute. How long will it take the fan to change the air in a room 27 ft. by 25 ft. by 10 ft.?" Now you all know what you are supposed to do: multiply 27 by 25 by 10 and divide the result into 3375. But the assumption behind this is that the room is hermetically sealed and that the fan evacuates all the air before any new air comes in! Absurd! This is not off by a little bit, it's off by maybe an order of magnitude. You *could* do a sensible discrete approximation to this by evacuating 10% of the air, replacing it with replacing it with fresh air and thereby diluting the old air, and repeating this process until the old air is no longer noticeable. That's a model that would make more sense. You obtain a linear recursion for the amount of "old" air that is left after k evacuations, and you ask how long it will be until the old air can no longer be perceived. This is a reasonable mathematical model; by my definition, the original word problem was not a good one.

A word of caution: there are word problems which were never meant to be taken seriously. The context is deliberately whimsical, and is intended to add lightness and humor to a heavy lesson. For example, Kolmogorov in 1966 gave the problem of a bee and a lump of sugar at two distinct points inside a triangle. The bee wishes to fly a minimum length path to the lump of sugar, under the condition that she must touch all three sides of the triangle along the way. I have no objection to such a problem—in fact, it's lovely! But nobody pretends it's about actual bees! What I object to are problems that pretend to be real but couldn't be.

Our second objection is that real modeling requires a lot of knowledge on the teacher's part, knowledge of a lot of fields outside of mathematics! That's true, but needs to be examined very carefully. Mathematics gets applied in all aspects of everyday life, intelligent citizenship, and other disciplines and occupations. Furthermore, most branches of mathematics, certainly all at the school and undergraduate level, have significant practical applications. In fact, there are unexpected and rather interesting connections between these two observations. When we worry that teachers, and students, may not know certain fields to which mathematics is applied, we often have in mind the field of physics. What mathematics is most applied to physics? Classical, continuous, analysis. Discrete mathematics is just as important for applications as continuous mathematics, and there tend to be many more applications to everyday life, operations analysis, and the social sciences, where the natural experiences of both teachers and students can give a great deal of guidance and insight. Thus discrete mathematics is an arena where we can bridge the gap between mathematical modeling in the classroom and mathematical modeling in the rest of the world with unusual effectiveness. What we are saying is that mathematical modeling can be particularly accessible when the resulting mathematical field at the heart of the development is in the area of discrete mathematics. Voting and fair division and the cleaning of streets are just as interesting mathematically as moments of inertia, and they use a lot of available intuition and experience.

Here are partial descriptions of some of my favorite modeling situations which lead to discrete mathematics and can be made accessible to high school students.

(a) Traditional private line pricing in the telephone business leads to minimal spanning trees, Cayley's theorem as well as Prim's and Kruskal's algorithms, Shamos' shortcuts, the Steiner network problem, and NP-completeness. The key modeling question: what is meant by "fair" pricing? This question drove much of the historical development. A discussion of the private-line pricing problem is given in [**7**].

(b) Building a counting circuit in a computer leads to the problem of enumerating Hamiltonian cycles for the graph which is the vertex and edge structure of an n-dimensional cube. It is easy to give an example of a single such Hamiltonian cycle, but how many different cycles are there? The graph theory soon becomes mixed with group theory. The key modeling question: when are two cycles "different"? It turns out that for engineering purposes—and this is where modeling is especially important—you want two Hamiltonian cycles to be *not* different (i.e., equivalent) if one can be obtained from the other by a symmetry of the n-dimensional cube. How many equivalence classes of Hamiltonian cycles are possible on an n-dimensional cube? The answer appears to be unknown for dimensions $n \geq 6$.

The mathematical formulation of this problem, and the complete discussion for four dimensions (i.e., counting from 0 to 15) may be found in [**3**]. This paper also relates the counting problem to the earlier Gray Code work during World War II, which was essentially a problem of analog-to-digital conversion. Martin Gardner refers to the answer for $n = 5$ in [**2**].

(c) In baseball, some of the modeling has been done for us, as in the definition of batting averages. If an additional hit takes a player's average from .299 to .306, how many at-bats and how many hits has that player had? This turns into a wonderful number theory problem, and involves Farey Series and continued fractions if we so choose. It is mathematical detective work: how do you turn a decimal into a fraction? We traditionally teach this for terminating decimals and repeating decimals, but not for arbitrary decimals known to a certain number of places—like batting averages.

The baseball example as such has not appeared in print; it is part of the author's lecture "Some Mathematics of Baseball" [**6**], which is one of the American Mathematical Society's videotaped "Selected Lectures in Mathematics". The same problem arises with free-throw percentages in basketball, and may be found in [**5**].

(d) Ed Gilbert at AT&T Bell Labs, who was involved in the research of (a) and (b), is the originator of the following problem: how do you build a perfect box? If you have six rectangular pieces of wood, what patterns of one piece covering another at an edge and at a corner are possible? There is some simple topology in this, and the Euler characteristic gives a lot of insight. Can you build a perfect box from six identical pieces of wood? The answer is "not in general", although it is possible if the dimensions of the blocks of wood satisfy certain conditions. Gilbert's article on this subject is [**4**].

(e) There are many well-known and more traditional problem areas that meet our requirements. I shall mention just one, that of coding theory. Noiseless coding, such as Huffman Codes, and group codes for the binary symmetric noisy channel, are two very accessible subjects. The combination of geometry, beginning group theory, and linear algebra at the beginning of group codes is especially appealing.

A nice exposition of the basics of group codes from just the point of view recommended in the previous paragraph may be found in [**9**]. Huffman codes at a level appropriate for high-school students may be found in [**8**]; the proofs related to Huffman codes specifically but not to noiseless coding more generally, may be found in the appendices. A more nearly complete exposition of noiseless coding appears, for example, as chapter 2 of [**1**].

Let us close with the third objection to mathematical modeling, namely the loss of certainty. There is personal judgment in the problem formulation parts (1)–(3), which is especially noticeable when, in (8), the results don't fit reality. Worse than that, there are honest differences of opinion; for example, if a problem concerns fair division, or an optimum location, what to one person looks fair may not seem fair to another. Or, to give another example, when competing criteria in an optimization problem are naturally measured in different units, such as lives and dollars, then there is no obvious way to equate them, and disagreement is inevitable. This contradicts the myth, held by many students and, alas, some teachers, that mathematics is a field of single right methods, single right answers, and unambiguous truth. This is actually not true of pure mathematics either, but it isn't even close when you apply mathematics to the rest of the world. We have to admit that this observation may be especially distressing to those who like mathematics primarily because it is a way of making a reasonable living and at the same time minimizing any danger of involvement with the real world. For such people, word problems are survivable, because of their degree of unreality, but mathematical modeling may cause great unhappiness. Their response may be to deny that modeling has a place in the mathematics curriculum. Now discrete mathematics is especially useful in applying mathematics in relatively controversial areas. Is this one of the reasons why its place in the curriculum has been hard to secure?

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