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Discrete Mathematics in K–2 Classrooms

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Introduction

This article describes two K–2 classrooms that I have observed and/or taught during the 1996-97 school year. Critics have claimed that mathematics taught in primary grades (K–2) is nothing more than memorizing facts, contains little content beyond computation, and that topics in discrete mathematics cannot be thoughtfully discussed by children at these levels. I strongly disagree. For the past ten years, I have been involved with professional development projects for K–12 teachers of mathematics, including the Leadership Program in Discrete Mathematics (see Rosenstein and DeBellis [7]). This experience, coupled with my background in mathematics education, has provided many opportunities to collaborate with K–12 teachers who are implementing discrete mathematics in their classrooms. Based on these experiences, I have come to believe that not only is it important to incorporate discrete mathematics into existing curriculum, but that K–2 classrooms are a natural place to begin developing the rudiments of the subject.

The current K–2 curriculum

Traditional K–2 mathematics curricula include topics such as counting, writing numerals, whole number operations (addition, subtraction, multiplication), fractions, estimation, place value, measurement, geometry, and problem solving. Within the past ten years, some curriculum developers have also included topics in probability and statistics for K–2 children which typically focus on making predictions about experiments and on recording and interpreting data. The following general summary of grade level expectations in mathematics is based on my review of several current K–2 mathematics curriculum guides from New Jersey public schools.

By the end of kindergarten, children should be able to count and write numbers up to twenty, as well as add and subtract these numbers. They

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should be able to measure in ad hoc units — for example, a desk may be three pencils long — and understand spatial relationships such as over, under, top, bottom, middle, left, right, inside, and outside. They should be able to identify planar figures such as a circle, triangle, rectangle, and square; and sort or classify objects by attribute — color, shape, or size. Children in kindergarten should also begin flipping coins and recording outcomes.

By the end of first grade, children should be able to count and write numbers up to one hundred and add and subtract two-digit numbers. They should begin to have some part-whole understanding of fractions and be familiar with fractional amounts such as 1/2, 1/3, and 1/4. They should be able to identify spatial figures such as a ball, cube, cone, can, and box, and be able to acquire information from pictures, text, and charts. They should be able to identify and discuss notions of symmetry and perimeter in a square, rectangle, triangle, and circle. They should be able to solve two-step word problems.

By the end of second grade, these same children should be able to count and write numbers up to 999; add and subtract three-digit numbers; know multiplication facts with 0, 1, 2, 3, 4, and 5 as factors; and write fractions symbolically and work with mixed numbers. They should also know the place value system for ones, tens, and hundreds; be able to make and use charts, tables, and drawings to solve problems; identify three-dimensional geometric shapes — cube, cylinder, sphere, cone, and rectangular prism; and discuss area and volume. It is also during the primary school years that children learn about systems: coins, clocks, calendars, maps, metric system, standard measurement system, bar graphs, and pie graphs.

"Young children enter school with informal strategies for solving mathematical problems, communication skills, ideas about how number and shape connect to each other and to their world, and reasoning skills. In grades K–2, students should build upon these informal strategies" (see the *New Jersey Mathematics Curriculum Framework* [**6**], page 83). Cognitively, according to Piaget, this population acquires knowledge through thought and action (see Inhelder [**3**]). As a result, mathematical concepts are taught through the physical manipulation of objects, through role playing, through story telling, and through thematic teaching approaches.

Existing curricula for the primary grades already include natural connections to discrete mathematics topics. For example, during the first marking period, many K–2 grades spend time classifying and sorting, including pattern detection (identify the pattern) and pattern projection (what comes next in the sequence). In fact, several K–2 textbooks which claim to include discrete mathematics topics simply include sorting activities and nothing more. Second graders spend time learning the fundamentals of geometry. The curriculum usually includes topics on shape, size, what defines an object, and what makes two objects different from one another. But I have also observed second grade children explain what makes a triangle, circle, and square the same. These children are capable of doing far more complex mathematics than we have traditionally expected. The following accounts serve to demonstrate what can be done in K–2 classrooms.

A visit to Grade 2

The day was "math day" (an entire day devoted to learning mathematics) when I visited Sharon Heil's second grade classroom at the Kossmann School in Long Valley, New Jersey. The school has roughly five hundred children in grades K through 2. Ms. Heil teaches in a self-contained classroom of twenty-four students. She described this class as a truly heterogeneous group, comprising students from both farm families and middle-management families. Academically, the students have a wide range of abilities; some students receive academic support in the resource room, others receive basic skills assistance in mathematics, language, and/or reading, and others are high-achieving, articulate problem solvers. In general, she feels all her students are enthusiastic learners and very curious about the world around them. In the classroom descriptions that follow, the names of the children are fictitious so that they remain anonymous.

Until participating in the 1995 Leadership Program for Discrete Mathematics (see Rosenstein and DeBellis [**7**]), Ms. Heil had not taken any mathematics courses since graduating from college over 20 years ago. To her credit, she is among many elementary school teachers who recognize the need to upgrade their own mathematical learning. It was not easy for her, but I witnessed the benefits — a teacher who provides thoughtful, meaningful mathematical experiences to her students.

Sharon Heil sat on a chair near a carpeted open space in her classroom. The students systematically pushed their desks to the side of the room and lined up near the chalkboard, silently waiting for instructions. Ms. Heil asked the children to randomly sit on the floor in front of her without any parts of their body touching one another. They were excited because they saw her holding a kickball and thought the idea of playing with a ball inside the building was neat! She told them that this is an activity where everyone is silent. "I'm going to give the ball to Annie. You must pass the ball from student to student (without throwing it or moving from your seat) so that everyone touches it at least once and gets it back to Annie."

Ms. Heil was imagining the children as vertices in a graph, where two children were joined by an edge if they were close enough to hand the ball from one to the other without changing positions. She was asking them to find a circuit which included all the children; soon she would ask them to find a Hamilton circuit. Needless to say, she had initiated this activity without introducing any of these terms. The children began to pass the ball to each other without talking. When it got back to Annie, the teacher said, "raise your hand if you touched the ball once." Sixteen children raised their hands. "Raise your hand if you touched the ball twice." Eight children raised their hands and the teacher asked them to stand. These eight children are pictured as Frank, Charlie, Zachary, Lisa, Deanne, Michael, Anthony, and

Annie in Figure 1. The ball was given to Janet who was the last person to touch the ball once (and was sitting on the floor) before giving it to Frank who was the first person to touch the ball twice (and was standing). The teacher asked, "is there a shorter ..." and was interrupted by Daniel who suggested that there was another way to pass the ball. He said, "Janet gives it to Jackie. Jackie gives it to Lisa and Lisa gives it to Annie." The teacher asked these students to pass the ball in this fashion to show that such a path was possible. After doing so, Ms. Heil asked Jackie, Lisa, and Annie to stand and all others to sit. Figure 1 indicates the two paths proposed by the children; the original path consisting of eight children who touched the ball twice (Frank to Charlie to Zachary to Lisa to Deanne to Michael to Anthony to Annie) and a shorter path (Jackie to Lisa to Annie) introduced by Daniel.

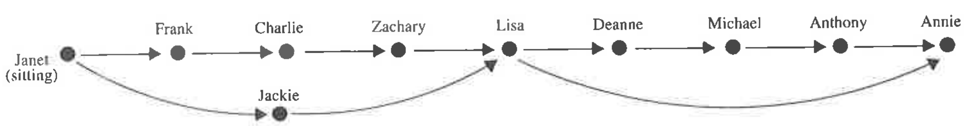

FIGURE 1.

A discussion ensued about how to make a shorter route. Lisa suggested that there is no route which leaves fewer than three people standing because, "how can you count one more person out? You'd have to throw the ball." Daniel insisted on a new proposal — Janet to Matt to Kenny to Maryann to Annie — then independently realized that this path was longer than his original three-person path. The group concluded that three was the fewest number of children who must touch the ball twice, until Daniel persisted that the ball can be passed with only two students touching it twice. He aggressively argued, "Janet to ..." but Cindy interrupted, "you can change the way you're passing the ball to only have Annie touch it twice." Daniel blurted, "you can just go in a circle." These suggestions happened simultaneously and the lesson that follows was crafted by a gifted teacher who encourages children to explain what they are thinking.

Ms. Heil interrupted to recognize appropriately the thoughtful comments that took place and asked, "Okay, let's consider individually what Cindy and Daniel each have said." The teacher stood and asked Cindy to exchange positions with her. All children were now sitting on the floor, waiting for their next instruction. Many children were laughing because they thought it was funny that the teacher was sitting on the floor, with her legs crossed like all the children, and Cindy was now in the teacher's position. Cindy began, "Okay, Annie gives to Lisa, Lisa gives to Robert, Robert gives to Sharon, Sharon gives to Mark, etc." Cindy orchestrated the movement of the ball in such a way that the only person who touched the ball twice was Annie. She worked outward from Annie, making sure that everyone touched the ball, but reserved a path of people along the front wall which she later used as the path that returned the ball to Annie. Her behavior was very similar to that of a mathematician as she or he works to find a Hamilton circuit in a graph; that is to say, each decision about to whom the ball should next be given is made keeping in mind that everyone needed to touch it once and "a last path" was needed in order to get back to the beginning. I thought to myself, "Am I really in a second grade classroom?"

The teacher stood after the task was completed and asked what just happened? The children explained, "If you do it the first way, the shortest way we could get is three people who touched the ball twice, but if you do it Cindy's way, you only get one person who touches the ball twice, Annie, so Cindy's way is shorter."

"Now, what about Daniel's comment. Daniel, what did you say before?" He replied, "You can just go in a circle." Ms. Heil suggested, "Okay everyone, let's get into a circle." From a theoretical perspective, the graph represented by the children has been changed, but from an educational perspective, Ms. Heil was presented with a valuable opportunity to take the lesson into uncharted territory, of which she quickly took advantage.

All the children sat in a large circle on the floor. The ball was given to Annie. "Now can you pass it so that everyone touches it once and it gets back to Annie?" The children passed the ball and when it was returned to Annie the teacher asked, "which way was easier?" They shouted, "circle!" Why? "Because you know where you're going." One child actually explained, "because you don't have to think about where to pass it next, you just get the ball from one side and pass it right to the next." This child was formulating a fundamental idea in computer science — that by arranging many individual units, each with a simple task, a large-scale, complex task can be performed. In computer-science terms, the children were simulating cellular automata.

Ms. Heil continued, "Is there any other way you could arrange yourselves so that ..." Another child shouted, "a square". The children arranged themselves into a square. They again passed the ball. "If I pass the ball along the square, is it similar or different if we pass it on a circle?" Several hands were raised immediately and the children responded, "similar." One child explained, "because we're still passing the ball to someone next to you." Another child shouted, "I think we should do a triangle because we could pass the ball there too." Ms. Heil said, "Good idea!" The class arranged itself into a triangle and passed the ball for a third time.

"So is the path in the triangle similar or different to the path in the square?" The class responded, "Similar." "What about the path in the triangle and the path in the circle?" "Similar." "What about the shape of the circle and the shape of the square?" "Different", they shouted. "What about the shape of the circle and the shape of the triangle?" "All their shapes are different." "Very good!" the teacher said as she looked at me in surprise. "So a path in a circle, square, or triangle is similar even though their shapes are different." This demonstrated that second graders are capable of understanding the rudiments of topological equivalence.

"What if we start with Annie, but don't end there? Could we arrange ourselves in such a way that the ball starts with Annie and everyone touches it exactly once but it doesn't have to end with Annie?" The students were still arranged in their triangle shape. They looked at each other as if this was too easy a question. One child said, "we don't have to move. Just pass it to Annie and end with Missy." Missy was the child who sat immediately to the left of Annie as the ball was passed to the right. Another child instantly shouted, "we could stand in a line." Ms. Heil began, "Okay, let's ..." and was interrupted by Daniel who said, "No, even if you stand in a line you get it back to the first person." The teacher and I were both confused. Did Daniel see a way for people to stand in a line and still make a circuit? Ms. Heil inquired, "What do you mean?" Daniel said, "You just have to give the ball to the first person, the first person gives it the third person, the third to the fifth, all the way to the end, and then that person just has to pass it back to the ones who didn't touch it yet." I was truly amazed at this second grader's insight.

Ms. Heil said, "Daniel thinks that you can get the ball back to the beginning if you stand in a line and everyone only touches it once except for the first person. Who agrees with Daniel?" A few hands were raised, tentatively. "Okay Daniel, show us what you mean." All twenty-four students stood in a straight line except Daniel. He gave the ball to Annie, who was standing at one end, and said, "Annie gives it to Frank, Frank gives it to Michael, Michael gives it ..." until the ball was passed back to Annie with everyone touching it exactly once. Figure 2 depicts a simplified version of the path that Daniel, a second-grade student, constructed in his mind; Daniel's path involved all twenty-three children.

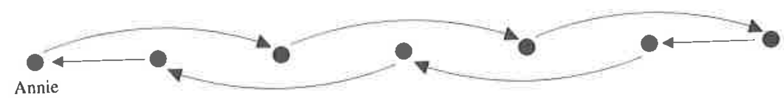

FIGURE 2.

"What just happened here?" the teacher asked. One child explained, "even though we're standing in a line you can get the ball back to Annie and only touch it once." The lesson concluded by introducing the words "path" and "circuit". When the teacher introduced the word circuit, Pete shouted, "is that like a circuit breaker?" Children make connections naturally if they're allowed to investigate their world. The words circuit and path were on the next's week spelling test.

I sat back in my chair in amazement. Second graders are quite capable of intuitively constructing paths and circuits in quite complex ways. They are able to recognize that a ball's path is the same in a circle, in a square, or in a triangle. Of course, they were unable to discuss graph isomorphism, but they found ways that a circle could be the same as a square and as a triangle. Further, they were able to maintain, at the same time, that these objects have different shapes in the Euclidean sense. They were able to identify and generate shorter paths — by finding a new way to pass the ball that would involve fewer children. A second grade classroom can fully engage in a dialogue which is filled with rich mathematical discourse.

This example shows how widely accessible topics in discrete mathematics can be. A young person's problem may be worded as follows: Given N children, randomly seated on the floor, how do we pass a ball so that each child touches the ball only once and so that it gets back to the first person who touched the ball? A similar challenge for a more mature problem solver may be worded a bit differently: Given N points in the plane — each connected by an edge with a few of its neighbors, find a Hamilton circuit. Essentially, both populations (children and adults) are able to discuss and solve these problems successfully. Having children think about such problems during their primary school years will provide a foundation for later mathematical development.

It might be said that this second grade classroom was full of gifted children, or at least Daniel (the child who generated many interesting paths during this lesson) was quite talented. Actually, none of the students in this class have been classified as "gifted", including Daniel. (In the Kossmann School, to be classified as "gifted" the child must score at least a 135 on the Wechsler Intelligence Scale for Children (W.I.S.C.).) Maybe educators need to evaluate how we determine if a child is mathematically talented. I saw a few children in this class who demonstrated powerful mathematical thinking and who I would classify as "gifted."

A visit to Kindergarten

I was invited into Michele Midura's classroom at the Irving Primary School in Highland Park, New Jersey to teach a discrete mathematics lesson. The school has roughly four hundred children in grades K through 2. She teaches a self-contained full-day kindergarten class with eighteen students, sixteen of whom were in attendance on the day of my lesson. She describes the class as developmentally, culturally, and economically diverse. Academically, the students have a wide range of abilities; some students have learning disabilities while others are reading and writing on a first grade level. Prior to my arrival, she informed me that the theme for the month was sports and nutrition and suggested that whatever math I did, I should somehow tie it to one of those themes.

In keeping with the "sports" theme, I decided to introduce the notion of a tournament by having each pair of children in a small group roll a giant die to determine a winner. I was uncertain how much of this topic kindergarten children would be able to understand, or even whether they would be able to determine if every player in their group competed against every other player exactly once. To see if they were capable of both enumerating all possibilities and knowing when they found all possibilities, I decided to begin with a combinatorics activity.

I entered her classroom with a large duffle bag filled with sports equipment, sneakers, and several two-foot long arrows made from poster-board paper. After their normal routine of hanging up coats, turning in homework, selecting hot or cold lunches, and telling their morning news (what Ms. Midura calls "show-and-tell"), I am able to introduce a math problem. The children were sitting on the floor arranged in a big square. I said, "Close your eyes! Keep them closed!" and reached into my bag. I pulled out a set of plastic bowling pins with two plastic bowling balls, still in their original wrapper. I asked, "What sport would you be playing if you needed these?" They simultaneously yelled, "Bowling!" Sixteen little people yelling an answer in unison caught me by surprise. Their level of excitement is infectious. This is nothing like teaching undergraduates! I placed the bowling pins on the floor in the middle of the square and said, "Close your eyes!" Several children began wiggling with anticipation. "Keep them closed!", I said. After I pulled out a tennis racket from the bag I asked, "What sport would you be playing if you needed this?" "Tennis!", they yelled. I placed the racket on the floor next to the bowling pins. We played the "close your eyes" routine two more times as I pulled out a pair of pink sneakers and a pair of Rebok sneakers and placed them on the floor.

"How many ways can you choose a pair of sneakers and a sport to play?", I asked. There was dead silence. I thought, "Uh, oh ... this is probably too hard." I regrouped and asked a different question, "Can anyone choose a pair of sneakers and a sport to play?" All sixteen children raised their hands. Anita chose the pink sneakers and bowling pins. I asked if anyone could find another way. Jimmy chose the Rebok sneakers and the tennis racket. Both children were standing in front of the class, wearing the sneakers they selected and holding their chosen piece of sports equipment. I pointed to each item and repeated, "Okay, Anita wears pink sneakers and bowls. Jimmy wears what?" The children together responded, "Rebok!", "and plays?", "Tennis!", they yelled. "Okay, can anyone find a different way to wear sneakers and play a sport?" Sean raised his hand, walked in front of the four items (now on the floor) and stared at them. After a few seconds, I asked if he would like a helper and I noticed Ms. Midura standing behind all the children nodding her head yes. Sean nodded his head up and down and picked the boy who was sitting next to him. Together they selected the Rebok sneakers (because pink sneakers were for girls) and the bowling pins. I repeated their choices, "Okay now we have a different way. Rebok sneakers and bowling pins. Can anybody find another way?" Cindy selected pink sneakers and the tennis racket. I asked, "What did Anita pick?" The students described her selection. "What did Jimmy pick?", "What did Sean pick?", "What did Cindy pick?" Each time the children described the choice of sneakers and sports equipment.

Now I returned to my original question, "How many ways can you choose a pair of sneakers and a sport to play?" "Four!", they yelled, continuing to respond in unison. "How did you know there were four?", I asked. One child explained, "because Anita's way is one, Jimmy is two, Sean is three, Cindy is four, and there's no other way to make a match that's different." "Okay, what are the different ways?" Together the children described each way of matching a pair of sneakers with a piece of equipment. Each time a new pair was mentioned, I placed a large fluorescent colored arrow on the floor to show the match. After all the pairs were found, we counted the number of arrow heads together and discovered four different ways. I said, "Close your eyes!" This was now a game for them. They're wiggling and getting excited because they know something else is coming out of the bag.

I placed a whiffle ball and bat on the floor next to the tennis racket, removed the arrows, and asked, "Now how many ways can we match a pair of sneakers with a piece of sports equipment?" The children enumerated all possibilities in a similar way described above. We again placed the arrows on the floor to show all six possibilities and count the arrow heads. "Close your eyes!" The children were now peeking (and telling me that they're peeking) and laughing as I pulled a pair of men's dress shoes from my bag. I asked them, "Do you know what these are?" No one responded. I said, "Geek sneakers!" and they all started laughing. "Now how many ways can we match a pair of sneakers with a piece of sports equipment?" A variety of children suggested simultaneously, "ten", "six", "nine", "twelve". I asked the young girl who responded "nine" to explain how she got her answer. After a bit of encouragement, she said that she counted every object on the floor (six individual sneakers and three pieces of equipment). Then one child yelled, "No, it's three plus three plus three." I am stunned by both responses.

I explained that mathematicians count all sorts of things. Sometimes we count individual things; for example, if we count all the sneakers and all the equipment, we find that there are nine things altogether. But sometimes we count groups of things, and that it is good to learn how to do both. Here we are counting how many ways there are of forming a group which has one pair of sneakers and one kind of equipment. I asked Carlo to use the arrows to show me what he meant by "three plus three plus three". He placed the arrows on the floor to show each possibility and then proceeded to count the arrow heads; altogether there are nine. He showed that each pair of sneakers can be matched with three different pieces of sports equipment and was able to describe this matching as "three plus three plus three". I reviewed this generalization with the children, and so ended the combinatorics activity.

Now convinced that kindergarten children will be able to determine if all players competed against one another in a tournament, I asked, "Find an X made from masking tape on the floor and sit on top of it." In advance, eighteen small X's were positioned in three circles (six to a circle) so that the fluorescent arrows could be placed between any two players to identify the winner and loser of that competition. All sixteen children, Ms. Midura, and her classroom assistant played in a tournament. Each group received one pair of giant dice and fifteen arrows of two lengths — nine long arrows (to be placed between players who are seated across the circle from one another) and six short arrows (to be placed between players who are seated next to one another). Player A rolled one die and Player B rolled the second die; together they determined the winner — the one who rolled the larger number — and placed an arrow on the floor pointing from the winner toward the loser. After all fifteen arrows were arranged according to the outcome of each competition, the children were asked to find a winning sequence — that is, a way of listing the six children in the group so that each one defeated the next one in the sequence; the winning sequence would be a Hamilton path in the directed graph.

In this lesson, it should be no surprise that the children were unable to find a winning sequence. I did several things wrong. First, the size of the groups (six) was too large for kindergarten children to even begin to look for a winning sequence. Having fifteen arrows pointing in many directions contained too much information for them to decipher. Second, the children were not comfortable with the arrows pointing to the loser; they wanted them to point to the winner and after several protests, the arrows pointed to each winner. Although this does not impede one from finding a Hamilton path, it does introduce another cognitive step in finding a winning sequence. Third, they really liked giant dice and I did not allow enough time for the children to "play" before I asked them to hold a tournament. Seasoned teachers of primary grades always allow time for play before they ask children to complete a task. Finally, I did not review the skill of finding a Hamilton path — that is to say, constructing a path along a series of directed edges so as to touch every vertex exactly once. I would certainly do this activity differently the next time, and expect that the children would find Hamilton paths; several kindergarten teachers in the Leadership Program have reported that they were able to do this with the children in their classes.

Although the lesson did not achieve what I initially intended, I learned that children in kindergarten are able to understand issues that arise in enumerating possibilities (sometimes an exercise in creative thinking) and determining that all possibilities have been exhausted. I also learned that children in kindergarten are quite capable of deciding who is a winner and how to position a directed edge to reflect that. I also believe that children at this age can identify a winning sequence in a tournament with four or five competitors, based on my observations of why they were unable to complete the task in the setting described here. In the long run, such activities help children develop strategies that are valuable for later use in problem solving, as well as for probability. Not to include such activities throughout the primary grades is a serious omission.

In this classroom I took a risk by attempting an activity which I had not tried before with children at this grade level. However, I have described this "failure" here so that teachers will understand that it is important for them to take similar risks in their classrooms. Teachers who attempt to bring discrete mathematics into their classrooms will find that sometimes their lessons are successful, and sometimes they are not. Nevertheless, they should continue to try out new activities, since ultimately their students will benefit, as they find ways of improving on their failed attempts, and even their present students will benefit, as the problem solving prepares them for future challenges.

A coloring example

A popular discrete mathematics topic among teachers who attend the Leadership Program is graph coloring. We introduce the topic by having groups of teachers work together to color a five-foot map of the United States. Each group is provided with two hundred circles cut from construction paper. There are ten different colors with twenty circles of each color. The initial question posed is to simply color the map. In a short time, some groups have nicely colored maps using all ten colors, other groups may be trying to use fewer colors, and yet others may define specific colors to represent characteristics of that state (i.e., all states that border the ocean, color orange or green). We now introduce the mapmaker problem. Imagine that you are a mapmaker and the cost to produce a map increases based on the number of different colors you use. Further, since every mapmaker wants individual regions to be clearly viewed on the map, no two regions that share a border may be colored with the same color. What are the fewest number of colors needed to color the map of the United States? Why? Can you identify areas of the map which cause problems? (If you haven't thought about this problem, take a few minutes to think about it before reading the rest of this article!)

When I walked into Sharon Heil's second grade classroom, I saw every piece of available wall space filled with students' work, from floor to ceiling. In the front of the classroom was a bulletin board dedicated to map coloring. There were several colored maps of the United States (partitioned into states) and several maps of Ohio (partitioned into counties, supplied by a Leadership Program participant from that state) which were colored by the students so that no two regions which share a border had the same color. I thought to myself, "I've seen that before ... but I wonder what second graders got from the experience."

Ms. Heil asked, "Does anyone remember what we did when we colored the maps?" Instantly, several hands were raised. One child explained that you "color two states with different colors if they're next to each other." "Anything else?", Ms. Heil asked. Another child explained, "We had to decide if corners counted or not." Several students proudly pointed to their map. Some groups decided that corners "counted", that is, they should be considered as shared borders, and some groups decided that they should not be considered as a common boundary. At the close of the discussion, Ms. Heil smiled and whispered, "we did that over one month ago."

This discussion was interesting because when we give the same exercise to K–8 teachers in our summer institutes, they too begin the activity with a similar struggle, namely, to decide if a point should be considered a shared boundary. An example of this occurs in the United States map at the point where Arizona, New Mexico, Colorado, and Utah meet. Most groups of teachers initially color these four states with four different colors. When they are asked to color their maps using the fewest colors, the issue arises as to what defines a border. Both populations (teachers and second graders) had to decide and define for themselves whether a point should be considered a shared border. All teachers were able to make the distinction that regions that meet at a point do not necessarily have to be considered a shared border; some second graders understood this, but others did not. When you make a decision that two regions that meet at a point may have the same color, the number of colors you will need to color a map may indeed be fewer. Hence some of the students' maps were colored with three colors, but others used more colors. Ms. Heil later explained that she did not focus on "fewest colors" and that the activity was intended to introduce coloring and discuss what makes something a border.

Map coloring is another example that shows how widely accessible topics in discrete mathematics can be; both adults and young children can engage in mathematical problem solving and experience similar difficulties. In the end they may resolve them quite differently, but they each have to define aspects of the problem that are not clear. Working through the "muck" of problem situation is one of the more difficult aspects of problem solving to teach; one must be willing to intimately engage the problem rather than passively perceive it (see Levine [4]). Introductory discrete mathematics topics seem to invite people from the non-mathematical community to think about their problems because difficult problems are easily understood and typically require little prerequisite knowledge; it may be that discrete mathematics can serve to attract under-represented persons into the mathematics field. It certainly attracts adults and young children alike!

Traditionally, coloring has been used in primary grade classrooms to help develop dexterity, creativity, and artistic talent. Now, with an introduction to discrete mathematics, teachers of K–2 classes can incorporate mathematics into their coloring book activities by asking the children to color the page so that no two regions that share a border have the same color. Once it is clear that everyone is able to color the page in this way, the teacher can introduce the question of using fewer colors. Before they begin, the children can be encouraged to talk about how they might develop a plan to use fewer colors. They can discuss why one color or two colors may not be enough to color the picture, or why N colors may be too many. These early conversations can help children begin to develop mathematical ideas about minimization and give practice in reasoning about lower and upper bounds. In addition, such lessons can also help develop powerful mathematical problem solving skills. For a good description of the value of coloring in K–4 classrooms, see Casey and Fellows [**1**].

Coloring books are also a good place to introduce children to constructing dual graphs, as part of an introduction to the topic of graphs. A typical page in a coloring book may look like the first bear in Figure 3. Primary

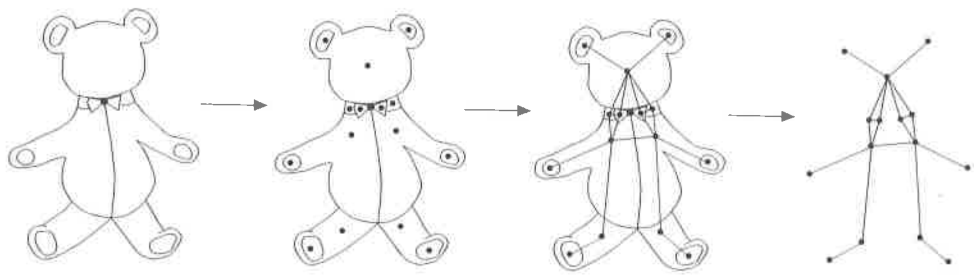

FIGURE 3.

grade children can be asked to place a dot inside each region and then connect two dots if the regions share a common border; the traditional kindergarten curriculum already includes learning notions like "inside/outside" and "next to" as geometry topics. These activities can provide a visual way to determine if children understand such notions. If the teacher chooses the picture wisely, she can then have further discussions with her students about the structure of the graphs. For example, in Figure 3, a teacher could ask the children to count the number of vertices and edges; to describe what parts of the graph look the same or what parts of the graph look different; to determine whether the graph is connected (together) or disconnected (in parts); or whether they can fold a piece of paper in such a way so that every vertex will lie on top of another vertex and every edge of the graph will lie on top of another edge. These experiences, if implemented in thoughtful ways, can help develop early notions of structure and symmetry.

The future of the K–2 curriculum

The K–2 curriculum for the twenty-first century needs to include technology topics and the mathematics that underlies computing. Some of these topics may be paths and circuits in graphs (consisting of vertices and edges), vertex coloring, Euler and Hamilton paths and circuits, shortest routes, counting, listing and sorting, and recognizing and using patterns in number and geometry. However, I am not suggesting schools should simply add more mathematical topics to an already packed curriculum. Rather, young children can learn to add numbers in the context of traveling along paths in weighted graphs (where each edge is assigned a "weight" which may, for example, be the distance between the sites represented by the vertices at the ends of the edge), or count "the number of ways" — an activity that can be done instead of just counting the natural numbers.

Primary grade students can establish efficient ways for dealing with their environment, and determine what makes something better (or shorter, or quicker) than something else. It is during this time that they can learn how to follow directions, follow classroom rules, follow a recipe, follow a map, and follow an algorithm. They can learn how to count the number of different ways to make change for one dollar, or how to systematically list the different ways to arrange three shirts with three pairs of pants. They can make flipchart storybooks to demonstrate the total number of outfits one can wear and, through such activities, come to know that mathematics is a way of thinking, not a way of memorizing. All of these topics are discussed in detail for the K–2 grade levels, as for other grade levels, in the discrete mathematics chapter of the *New Jersey Mathematics Curriculum Framework* (see Rosenstein, Caldwell, and Crown [**6**]).

At the primary grade levels, children can also be assisted and encouraged to come to understand what it means to be a powerful problem solver. A powerful problem solver is one who knows more than just a bunch of good strategies for solving a problem; it is a person who (among other things) uses intuition, generates conjectures, is creative, and perseveres. Young children can learn how to make a good prediction, how to remain comfortable even if a problem is left unsolved for several days, and that sometimes good problem solvers get wrong answers. They can also learn that working on a hard mathematical problem is sometimes frustrating, but that negative emotions can be regulated by the problem solver to a useful purpose (see DeBellis [**2**]). The ability to be successful at problem solving is no longer a higher order thinking skill that only mathematically talented children are expected to demonstrate; rather, all citizens of the twenty-first century will need this skill to function in a high-tech world. Today's kindergarten children will graduate in the year 2009.

Conclusions

I was quite surprised at the sophistication with which primary grade students can behave as scientists. As I walked around K–2 classrooms, observing other activities as well as those described in this article, I watched young children make conjectures, argue with team members for particular outcomes, demonstrate the ability to collect and record data accurately, verify that an experiment was run correctly by making sure the sum of each component equalled the total number of experiments conducted, and demonstrate the ability to make the distinction between a prediction and a best prediction. They also intuitively discussed fundamental notions of isomorphisms, algorithms, and topological equivalence. They were proud of themselves when each mathematical task was completed, just as the teachers were who worked on the same (or similar) problems in the Leadership Program.

Certain ideas in mathematics –– such as "isomorphism", "enumeration" (systematic listing of possibilities), or the ability to generate global complex behaviors with simple local rules — are very important and should be developed in young children. It should not be that these discussions happen in K–2 classrooms by chance. Mathematicians, mathematics educators, and teachers need to collaborate to define what "big mathematical ideas" ought to be learned at each grade level. Technology will continue to evolve and new mathematical discoveries will unfold. Unless school systems allow for the constant infusion of new mathematical topics and information into their curriculum, they will forever be teaching archaic topics at inappropriate grade levels.

Finally, K–2 discrete mathematics topics, when introduced by good teaching methods, can serve not only to build the foundations for important mathematical ideas, but also can serve as a vehicle to help cover traditional curriculum topics. K–2 teachers need continued support from university and college faculty members who are both knowledgeable about the content and understand the mathematical development of young children. At the same time, teachers need to remain active in the learning of mathematics, at whatever level is appropriate for them. It is only when teachers themselves are active problem solvers who, for example, think about problems they cannot yet solve, that they can model the desired mathematical behaviors for the children in their classes. Such activities and collaborations can only benefit the children.

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