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Discrete Mathematics Activities for Middle School

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There is a body of knowledge that has come to be known as discrete mathematics and much of it is accessible to middle-school students. Many related topics can already be found in the existing curriculum and others can be readily integrated into it. Discrete mathematics problems tend to be simply stated and easily motivated. They offer a rich, new source of diversified problem-solving experiences that range across all ability levels. Furthermore, they serve to portray mathematics from a broader perspective than many typical practice exercises.

It is equally important to note that problems in discrete mathematics can be incorporated into many of the hands-on activities that already are part of the established classroom scene. This article focuses on that connection through the two central ideas of *counting* and *change*. Counting is viewed through number patterns, computation, manipulation, and visualization, and these are connected to change through the mathematical idea of *iteration*. It is the notion of iteration — arithmetic, algebraic, and geometric — that brings alive the subject of mathematics, and it is through hands-on activities that it is made real. Emphasizing this combination when we teach offers a dynamic view of the discipline so greatly needed by today's middle school students.

This article begins with a sampling of discrete mathematics activities arising from a simple counting problem involving paper folding, then moves through others that can be analyzed by graphs, and ends with some applications of iteration through geometric transformations. The examples illustrate the importance of both content and pedagogy and show how discrete mathematics can be designed and woven into the broad fabric of middle-school mathematics.

Counting

Almost every middle school student and teacher has, at one time or another, used the folding of paper to explore a mathematical relationship.

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This first illustration shows some different ways one simple paper model can be tied into the arena of discrete mathematics.

Cut out some 2x8-inch strips of paper, one for each student. Have them fold the strips in half and in half again as shown in Figure 1. Let them visualize in their mind what the strip would look like unfolded.

FIGURE 1. Folding a strip of paper

Ask the students to mentally count all the rectangles that they visualize, including the squares. After writing their individual answers, let them compare and discuss their answers with other students. Once an agreement is reached in their groups, they can unfold the strips and check their answers by actually counting from the model. Finally, as a writing activity, have your students describe the algorithms they used for their counting, both in the abstract and in the concrete case.

This activity is much more than just one of visualization. It involves analysis and systematic counting. One approach might be to letter the squares (as in Figure 2a) and make a list of the 10 different rectangles using successive letters, four, three, two, and one at a time (Figure 2b). Another approach might be to show the solution in a graph with 4 vertices and 10 edges (Figure 2c). Six edges connect different vertices, denoting different starting and ending squares. Four edges connect vertices to themselves, indicating the same starting and ending square.

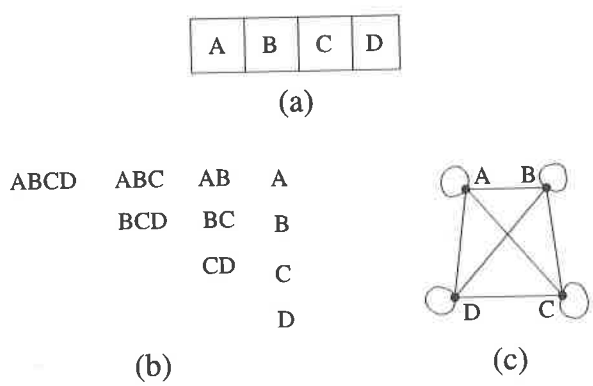


FIGURE 2. (a) An unfolded and labeled piece of paper. (b) Systematic listing of rectangles. (c) Using the edges of a graph to represent starting and ending squares of each rectangle.

The list reveals that, for two successive folds, the answer is 10, the sum of the first four counting numbers. Compare the number 10 for two folds to the number 3 for one fold:

Folded once: $1 + 2 = 3$.
Folded twice: $1 + 2 + 3 + 4 = 10$.

Ask your students to fold the strip in half a third time and ask for an educated guess as to how many rectangles will be in the unfolded strip now. See how many students can find and extend the pattern.

Folded three times: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.

Some middle school students may want to explore this problem further and look for a general solution. For n successive folds, the number of rectangles is the sum of the first 2^n counting numbers.

Folded n times: $1 + 2 + 3 + 4 + \ldots + 2^n = 2^{n-1}(2^n + 1)$.

Given the formula, this paper-folding activity now offers students an additional important experience with exponents and other algebraic symbolism. For example, with four successive folds, there are 136 different rectangles, since with $n = 4$,

$$2^{n-1}(2^n + 1) = 2^3(2^4 + 1) = 8(16 + 1) = 8(17) = 136.$$

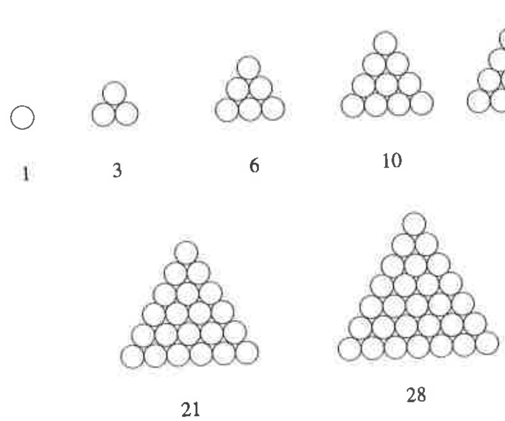
Are the counting numbers that come from this paper-folding activity, such as 3, 10, 36, and 136, special in any other way? You may recognize them from another discrete mathematics topic already in the middle school mathematics curriculum. They are members of the set of triangular numbers.

Figure 3a shows the triangular arrays which account for the name "triangular numbers". Triangular arrays such as these can be easily built and vividly displayed on an overhead projector. Figure 3b shows how the triangular numbers are calculated by summing the rows of the triangular arrays.

Another feature of the triangular numbers emerges if we look at a difference table. In a difference table, we first record the differences between successive triangular numbers — these are called "first differences". Then we record the difference between successive first differences — these are called "second differences." For the triangular numbers, second differences are all 1, as in Figure 3c.

Compare this to the familiar square numbers where the second differences are all 2, as in Figure 4. Here we see another topic from discrete mathematics, finite differences, closely connecting to the existing middle school curriculum.

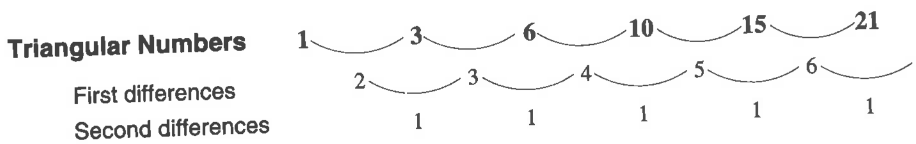
We can also look at other geometric arrays — squares, pentagons, hexagons, etc. — and introduce other sequences of "figurate numbers" — square numbers, pentagonal numbers, hexagonal numbers, etc. These geometric arrays lead to counting activities, number patterns to explore, discoveries



(a)

$$\begin{aligned}
&1 &&= 1\\
&1+2 &&= 3\\
&1+2+3 &&= 6\\
&1+2+3+4 &&= 10\\
&1+2+3+4+5 &&= 15\\
&1+2+3+4+5+6 &&= 21\\
&1+2+3+4+5+6+7 &&= 28\\
&\vdots\\
&1+2+3+\ldots+n &&= n(n+1)/2
\end{aligned}$$

(b)



(c)

FIGURE 3. (a) The triangular numbers. (b) Calculating the triangular numbers. (c) Difference table for triangular numbers.

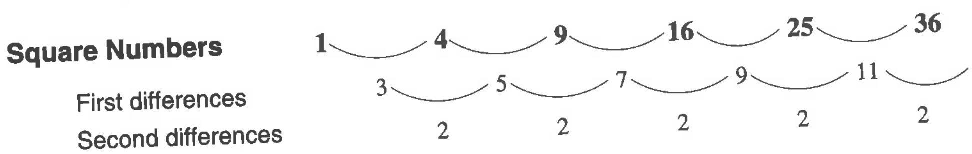


FIGURE 4. Difference table for the square numbers.

to make and test, and more questions worth investigating. For example, will pentagonal numbers have successive second differences that are all 3? For hexagonal numbers, will the successive second differences all be 4? The answer is yes for all figurate numbers of this type. In fact, any second degree, quadratic expression such as $n(n+1)/2$ must have constant second differences, an idea worth challenging your students to explore as a calculator activity.

Let us go back to the folded strip of paper for some more counting activities. Have your students label the squares on one side with the digits 1, 2, 3, and 4. Tear apart the four squares and the students have a nice model for some counting problems.

One good question is the following: how many different four-digit numbers can be formed using the squares? Let students work in teams arranging the digits, making lists, finding and applying counting procedures, and writing about their methods. This can lead nicely into the topics of permutations and factorials, since many students will discover that the answer 24 is expressed as $4 \times 3 \times 2 \times 1$. You can also ask how many numbers with 4, 3, 2, and 1 digits can be formed. The answer here is

$$(4 \times 3 \times 2 \times 1) + (4 \times 3 \times 2) + (4 \times 3) + 4 = 24 + 24 + 12 + 4 = 64.$$

For students at a higher level, ask them to turn the strip over and put the digits 5, 6, 7, and 8 on the back, with the 8 behind the 1, before tearing the squares apart. Ask the same two questions noted above. Here the algorithmic thinking is every bit as important as the numerical answers of

$$8 \times 6 \times 4 \times 2 = 384 \text{ and } (8 \times 6 \times 4 \times 2) + (8 \times 6 \times 4) + (8 \times 6) + 8 = 632.$$

For those who want a real challenge, label both sides of the strip 1 through 8 as noted above, but don't tear the strips apart. Folding only on existing creases, how many different numbers with 1, 2, 3, and 4 digits can be formed? At this level, some difficult analysis is called for by the students. Let them discuss and explore the problem mentally before they start folding and forming numbers in their hands with their paper strips.

On another day, review these results and then offer a variation. Mark the four squares of a newly folded strip of paper with the digits 1, 2, 3, and a decimal point (as in Figure 5). Separate the squares and think about possible arrangements using one or more of the squares. What are the different decimal numbers that can be formed?

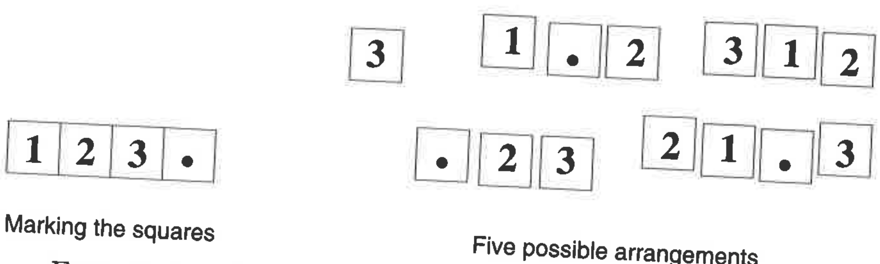


FIGURE 5. Creating decimal numbers with digits 1, 2, 3, and ".".

Counting the different possibilities can be an interesting and challenging activity for the middle-school student. But couched in the form of a class game or competition, much more classroom excitement and enthusiasm can be generated. Middle school teachers from the Rutgers University Leadership Program in Discrete Mathematics and others have related back to me several game variations they have used in their classes with great success. One of the more interesting formats uses no paper other than the strips used to introduce the activity. Every day from there on, begin the class with a number from the set, say 2. See how far you can get around the class, asking each student for the next larger decimal that can be formed from the set, before a mistake is made. When one occurs, stop. The following day, try again. Challenge the students to get through to the largest decimal without any mistakes. This may seem easy, but experience proves otherwise. The first few correct choices, in order starting with 2, are $2, 2.1, 2.13, 2.3, 2.31, 3, 3.1, 3.12, \ldots$

From the point of view of mathematical content, this activity deals with the important skill of ordering decimals. But even more important, students must create them, and to do so requires the ability to play freely and imaginatively with numbers and shapes in situations involving discrete choices. This skill needs to be developed and nurtured thoroughly in the middle grades by embedding it within the existing curriculum and around familiar classroom experiences. These kinds of simple exercises, while both fun and challenging for students at this age, lay the foundation that will enable them, in later years, to approach more profound and intriguing applications.

Graphs

Many problems can best be approached through models in the form of graphs. Graph models offer a kind of organizational structure that can be utilized in many problem-solving experiences involving both manipulatives and counting. Let us look at an example.

> *Five cubes of different colors are arranged in a row. How many different arrangements are possible?*

Many students familiar with counting know this is a permutation problem and know the answer to be $5! = 120$. But, when asked for an explanation or meaning, they have little to say because they really see nothing. Early counting experiences of this type need to be done with concrete materials and modeled in diagram form for better understanding. In the following example, we use five blocks, one of each of the colors green (G), orange (O), red (R), yellow (Y), and blue (B).

You might begin by arranging the cubes in a row and discussing their order. Have students suggest and show other orderings. Put the cubes in your hand and ask how many choices there are for the first position. How many choices remain for the second, and then the third, and the fourth, and the fifth positions? Connect these questions to the blocks and to the diagram in which the numbers are entered one at a time (see Figure 6), and to their product.

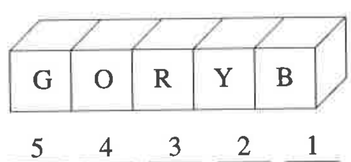


FIGURE 6.

A systematic listing of all solutions is often accessible and useful in solving many counting problems. However, a listing of the 120 choices here seems a bit tedious. This is one place where a graph can be useful. The vertices represent the cubes and the edges show all the possible connections (see Figure 7a). Every one of the 120 possible arrangement of the cubes is a distinct, directed path of four edges connecting the five vertices. The arrangement GORYB can be represented by a path as shown in Figure 7b.

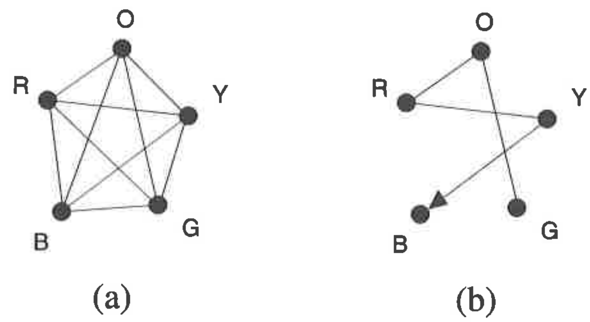


FIGURE 7. (a) Graph with vertices representing blocks. (b) Directed path representing the linear arrangement of blocks, GORYB.

Many good counting questions can be asked. How many of these paths start at G? How many start at G and end at B? How many have G next to B? How many do not have G next to B?

Situations can be analyzed and answers can be found from the graph. Have students trace out paths for given arrangements and arrange the cubes for given paths. (These require very different skills.) Have students count the number of edges in the complete graph and explain what the number means. Connect the answer to the problem of choosing two cubes from the set of five. See if they recognize the answer as a triangular number.

Many discrete mathematics problems are already in the textbooks and other available literature as examples addressing teaching methods or classroom issues. The EQUALS project at the Lawrence Hall of Science at the University of California at Berkeley, through its publications, *Get It Together*, suggests an interesting cooperative learning activity similar to the one just described. It is an arrangement problem involving six colored cubes.

Four students independently receive critical information, that they alone possess, about the arrangement. All students must participate because each student has information to contribute and needs to do so at the right time. The task is to arrange the six colored cubes in a row in the correct order.

a: Green is not next to yellow and purple is not next to green.
b: Orange is not next to yellow and green is not next to blue.
c: Yellow is not next to red, blue not next to purple, and red not next to orange.
d: Purple is not next to yellow, blue not next to orange, and green not next to red.

The problem offers an excellent example of a cooperative learning situation in the arena of discrete mathematics. One approach is hands-on, with the solution emerging through the arranging and rearranging of the colored cubes. Another approach is to draw a complete graph with 6 vertices representing the colors and 15 edges representing all possible ways any two colored cubes might touch each other, when arranged in a row. Clearly, in any given arrangement of the cubes, only some of these connections will be made. One by one, the students remove those edges not allowed by the restrictions they were given. In all, 10 edges will be eliminated from the graph. The 5 edges that remain reveal the only possible sequence, ordered left-to-right or right-to-left, shown in Figure 8.

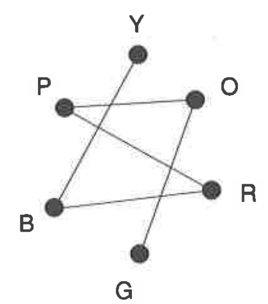


FIGURE 8. This path shows the only two possible arrangements of the six cubes when placed in a row: YBRPOG or GOPRBY.

Encourage students to make up similar sets of conditions on their own. Let them check one another's suggestions. Have them describe algorithms for creating problems that will ensure unique solutions. These are some of the important components of the critical thinking required for doing mathematics.

How can discrete math problems such as these, involving the ordering of colored cubes, be modified to assign lengths to the edges? Suppose, instead of having five colored cubes, teams of students select five whole numbers in the range 0 to 100. Imagine the numbers as the names of cities which are connected by airplane flights.

Begin by having the teams arbitrarily place their five vertices, identified by their choice of numbers. Next, have them assign distances to the edges corresponding to the differences between the numbers on the connected vertices representing cities. Here are some possible investigations to consider.

- Try to find the shortest route connecting all five cities. Where would you start and where would you end? What about a round trip that takes you through all five cities?
- Where would you start and end for the longest route, without repeating any connections? Is the same sequence the best for the longest round trip?

Have the teams try to find algorithms for solving these problems. Ask whether their procedures would change for an even instead of an odd number of cities. In middle school, students need the experience of exploring, trying, testing, and expressing their ideas in situations like these as much as they need to learn and apply known algorithms from discrete mathematics.

Figure 9 shows a complete graph, weighted on the edges by the distances for the five cities numbered 6, 32, 19, 84, and 61.

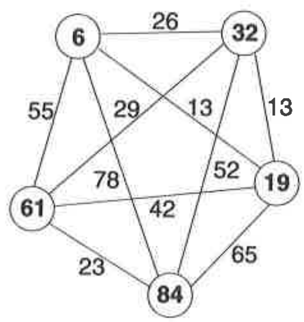


FIGURE 9. Graph showing five cities with distances.

There are 5! = 120 directed paths that connect the five vertices and 4! = 24 directed trips through them back to the starting point.

Finding the shortest paths and circuits through the five vertices in this situation does not require a great effort, especially if one realizes that tours among the points on the graph correspond to routes along the real number line. Finding the longest paths and circuits requires more thinking and testing. Searching for appropriate algorithms for any set of vertex values poses some interesting challenges.

Iteration

When the dynamics of change is built into a hands-on activity for the mathematics classroom through some iterative process, the experience becomes all the more powerful. One reason is that numerical, geometric, and algebraic relationships and connections often emerge from a single experience, as in the following activity.

Start with an equilateral triangle cut from paper. Mark a vertex P and repeat the following folding procedure through several stages:

> *When the vertex P appears in a triangle, fold it to the midpoint of the opposite side and then unfold.* (See Figure 10).

The outline of the folded paper at each stage is a trapezoid, but these trapezoids change through successive stages. How are they changing? What do you see?

From a measurement point of view, the trapezoids are growing in height. Start with a triangle whose area is 1 square unit and watch the areas of the trapezoids change.

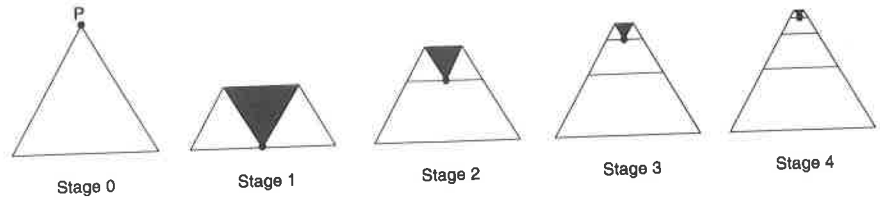


FIGURE 10. A trapezoid folding activity

The first triangle folded over has an area of $1/4$.
The second folded triangle has an area of $(1/4)^2$ or $1/16$.
The third has an area of $(1/4)^3$ or $1/64$, and so on.

Subtract these successive powers of $1/4$ from the original area of 1 to find what area remains for the trapezoid at each stage:

$$\underset{\text{Stage 1}}{\tfrac{3}{4}} \longrightarrow \underset{\text{Stage 2}}{\tfrac{15}{16}} \longrightarrow \underset{\text{Stage 3}}{\tfrac{63}{64}} \longrightarrow \underset{\text{Stage 4}}{\tfrac{255}{256}}$$

What else is changing as the process is repeated over and over? The unfolded stages reveal other interesting patterns of a discrete nature, as in Figure 11.

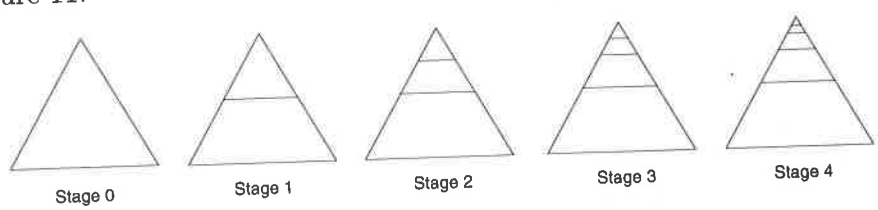


FIGURE 11.

In this form, we can view triangles and trapezoids in quite a different way, as shown in this table:

Stage	0	1	2	3	4	n
Number of triangles	1	2	3	4	5	$n+1$
Number of trapezoids	0	1	3	6	10	$n(n+1)/2$

Here again, we find the triangular numbers embedded in a counting problem centered around a geometric activity. Looking at the folds themselves, still another vision may appear. Let your students describe what they see.

One image is that of a strangely distorted ladder. When you climb it, each successive step is half as high and each successive rung half as wide. When you look up, you forever see reduced versions of exactly what you saw before. And the climb, step-by-step, is endless!

You can quickly see how some more powerful notions, such as perspective in art and limits in mathematics, can be brought into play. Students need to see, think, and talk about concepts such as these from an intuitive point of view during the middle school years. By choosing a good visual model and asking the right questions, one can bring together a host of related mathematical ideas in a single activity. And it is not surprising that many of these turn out to be discrete in nature.

Suppose the folding process is changed a bit, as described in the following algorithm.

> *Every time you have a triangle, fold each vertex to the midpoint of the opposite side. Cut off the corners and keep only the middle triangular piece at each stage.* (See Figure 12.)

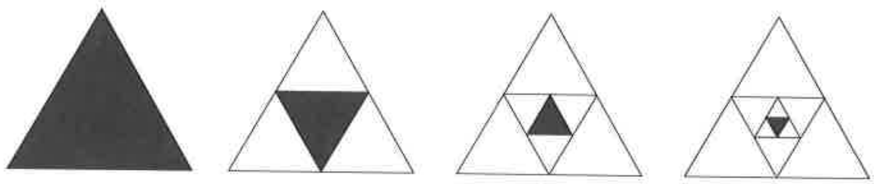


FIGURE 12.

A new set of figures is generated and new sets of number patterns emerge.

Stage	0	1	2	3	4	n
Number of triangles	1	1	1	1	1	1
Area	1	1/4	1/16	1/64	1/256	$(1/4)^n$
Perimeter	1	1/2	1/4	1/8	1/16	$(1/2)^n$

By interchanging what is kept and discarded in the folding and cutting process, an entirely different sequence of figures is created, as shown in Figure 13. This time, keep the corner pieces and discard the middle piece at each stage with each triangle. Now the process leads to an entirely different structure, a *fractal* called the Sierpinski triangle.

Stage	0	1	2	3	4	n
Number of triangles	1	3	9	27	81	3^n
Area	1	3/4	9/16	27/64	81/256	$(3/4)^n$
Perimeter	1	3/2	9/4	27/8	81/16	$(3/2)^n$

As an alternative approach in the classroom, have your students draw these two sets of figures on triangular dot paper. Choose a large triangle where the dots divide the sides into units that number a power of 2. This way the spacing of the dots will facilitate drawing several repeated reductions by one-half. For many students, both types of activities would be worthwhile. Indeed, seeing, drawing, and visualizing experiences all need to occur more

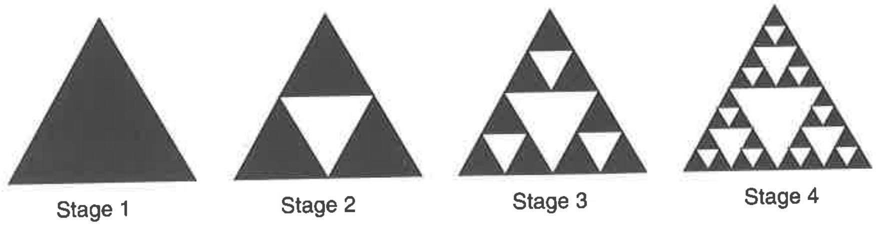


FIGURE 13.

often in the mathematics classroom to improve our students' abilities in visual literacy.

The distinct, discrete stages of growth clearly show an underlying property of fractals, that of *self-similarity*. Copies of the figure appear within itself at all scales. Three reduced images of the initial stage can be seen in stage 1. Three reduced images of stage 1 can be seen in stage 2. Three reduced images of stage 2 can be seen in stage 3, and so on.

The intricate structure of the emerging fractal can be measured by its *fractal dimension*. For the Sierpinski triangle, this complexity measurement is approximately 1.58. See Volume 1 of [**2**] for an introduction to the topic of fractals.

Is there an underlying structure here that is independent of the shape of the initial figure? That is, if we start with a different figure and repeatedly put together three copies of the figure, scaled to one-half, what do we get? Have students explore this question starting with other figures, such a right triangle, a scalene triangle, or even a square, rather than an equilateral triangle.

Start with a square cut from paper. Cut it in half vertically and horizontally. Use the rebuilding process shown in Figure 14 with three reduced copies at each stage placed in the shaded cells.

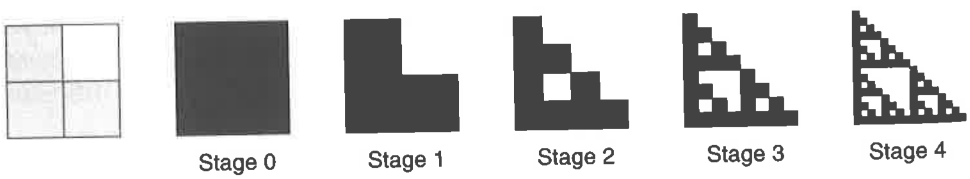


FIGURE 14. Iteration based on a square

It does not take many stages to see a familiar shape emerging. Have your students think, talk, and write about the similarities and the differences between the changing structures being generated from squares and those that were generated above from triangles. In both cases, of course, the limit structure is the Sierpinski triangle.

As a final activity, have students put their own personal twist to the rebuilding step in the iteration process, which can be abbreviated as *Reduce, Replicate, and Rebuild.*

Mentally label the three cells A, B, and C, as shown in Figure 15. When the reduced images are dropped back into the appropriate cells, consider possible rotations. In the sequence of figures shown in Figure 15, the reduced copy in cell A is always rotated 270° clockwise at each stage. Those copies placed in cells B and C always remain in their original orientation, which, for convenience, can be called a rotation of 0°.

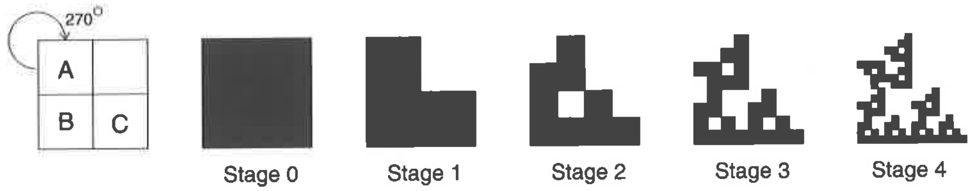


FIGURE 15.

Four choices of rotation are possible for each of the three cells. That gives $4 \times 4 \times 4 = 64$ different rebuilding codes using rotations. This can lead to the exploration of a whole family of related fractals with many different structures. Have students create their own personal fractals by making individual choices of rotations for cells A, B, and C. They can cut out and tape together their images or draw the first few stages on graph paper. The first four stages can be readily drawn using 2×2-inch initial squares on 1/8-inch graph paper.

When reflections are considered, another four transformations of the square can be explored. See Figure 16.

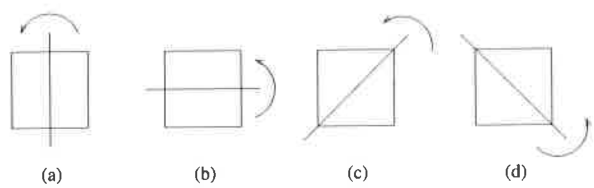


FIGURE 16. (a) Horizontal reflection about the vertical axis. (b) Vertical reflection about the horizontal axis. (c) Reflection about the lower-left, upper-right diagonal. (d) Reflection about the upper-left, lower-right diagonal.

In the sequence of iterations shown in Figure 17, the reduced copy in cell A is reflected about the upper-left, lower-right diagonal at each stage. Those copies in cells B and C remain in their original orientation.

In all, four rotations and four reflections can be made in each of the three square cells. With eight transformations possible in each cell, there must be $8 \times 8 \times 8 = 512$ different rebuilding codes. Will all 512 different building codes produce different fractals? The answer is no. Because of symmetry, some images will be duplicated. How many distinct fractal images will there be? The question is left for the reader to investigate and answer.

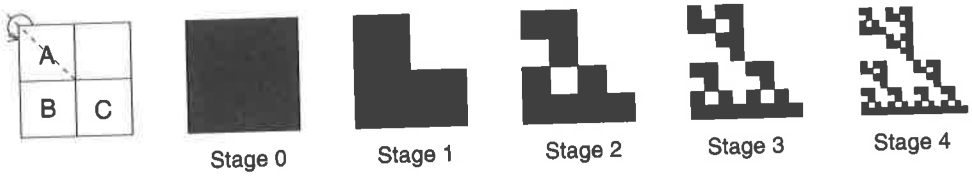


FIGURE 17.

Here we are, answering and asking yet another counting problem emerging from an iterative, geometric activity. The middle school curriculum is fertile ground for increased attention to situations involving discrete mathematics. The problems are all around us if we but look for them.

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