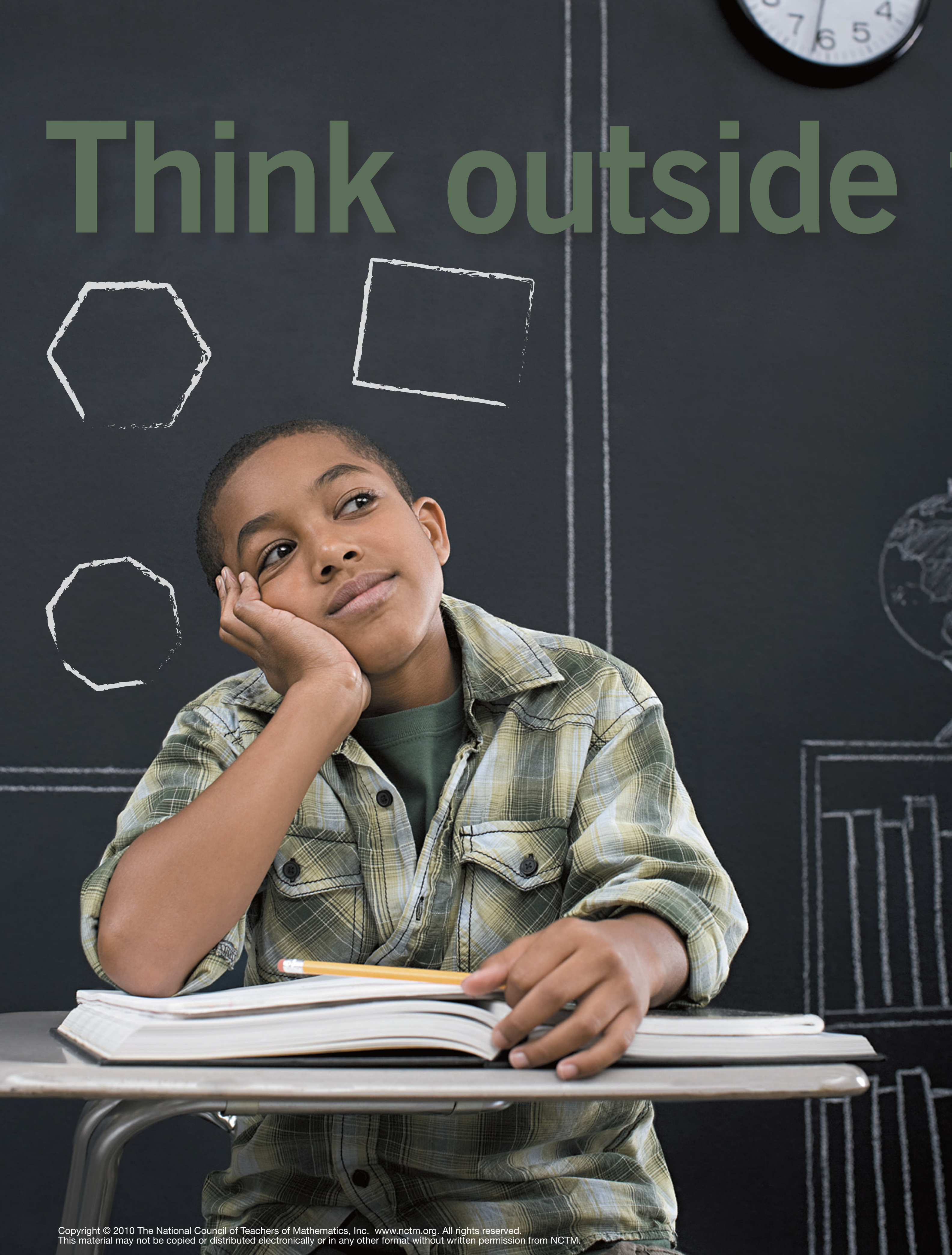
Think outside

the Polygon

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After constructing irregular polygons, students use geometry software to explore rotations and reflections and build spatial-visualization skills.

A new school year is beginning, and a new geometry unit is being introduced for the first time. I am always intrigued to see which of my students will have trouble with spatial visualization, especially those who have not struggled in previous mathematics strands.

Students appear to be able to see number relationships and patterns but have difficulty recognizing the visual properties of shapes, especially if the shapes are in different positions. Their difficulty in the visual and spatial realm is often linked to a lack of drawing experience and possibly undeveloped fine-motor skills. I would watch their painstaking movements as they tried to trace polygons only to realize that their work could not be used because the shapes were unrecognizable.

Seeing these frustrated students motivated me to develop a geometry project using The Geometer's Sketchpad™ to help eliminate the roadblocks to geometry for these students. For the spatially unchallenged students, using geometry software, such as Sketchpad, could only deepen their skills. (It should be noted that GeoGebra, a free software program available to all schools, can also be used for this project.)

Before starting this activity, our fifth-grade geometry unit explored polygons and their properties, angles (estimating and measuring), and symmetry. In discussing symmetry, we used plastic polygons to discover reflection and rotation, with lines and points of symmetry appearing *within* the polygons. Students categorized these polygons as having reflectional symmetry, rotational symmetry, both, or neither (see **fig. 1a**).

It was time to expand those parameters. I took one polygon, a scalene right triangle, that had been in the "no symmetry" category and demonstrated that it could indeed exhibit symmetry. If I drew a line of symmetry on or outside the polygon, a duplicate polygon could be created using reflection. Likewise, I could determine a point on or outside the polygon to create an infinite number of rotated polygons (see **fig. 1b**). Using this figure, I could produce a picture that included symmetry.

This instruction represented a paradigm shift in the students' thinking about symmetry, and I had to allow them time for all the information to sink in. Would these rules hold true for all polygons previously categorized as having no symmetry? I held up several paper cutouts of unusual polygons and demonstrated that even abnormal-looking polygons could become symmetrical if a duplicate was made and flipped over a line or turned on a point on or outside the polygon.

Students' first exposure to Sketchpad went smoothly. After a basic introduction to the toolbar functions and some time for exploration, we were ready to use it. As I pulled up a coordinate grid on the screen, I saw some puzzled looks: "*Graphing? We did that weeks ago. I thought we were doing geometry now!*" I take every opportunity to overlap and connect mathematics strands. To answer the unspoken thoughts behind those bewildered faces, I said that by combining these two mathematics areas we would be doing a special kind of geometry called *analytical geometry*.

Fig. 1 Students' early understanding of symmetry led them to categorize various polygons.

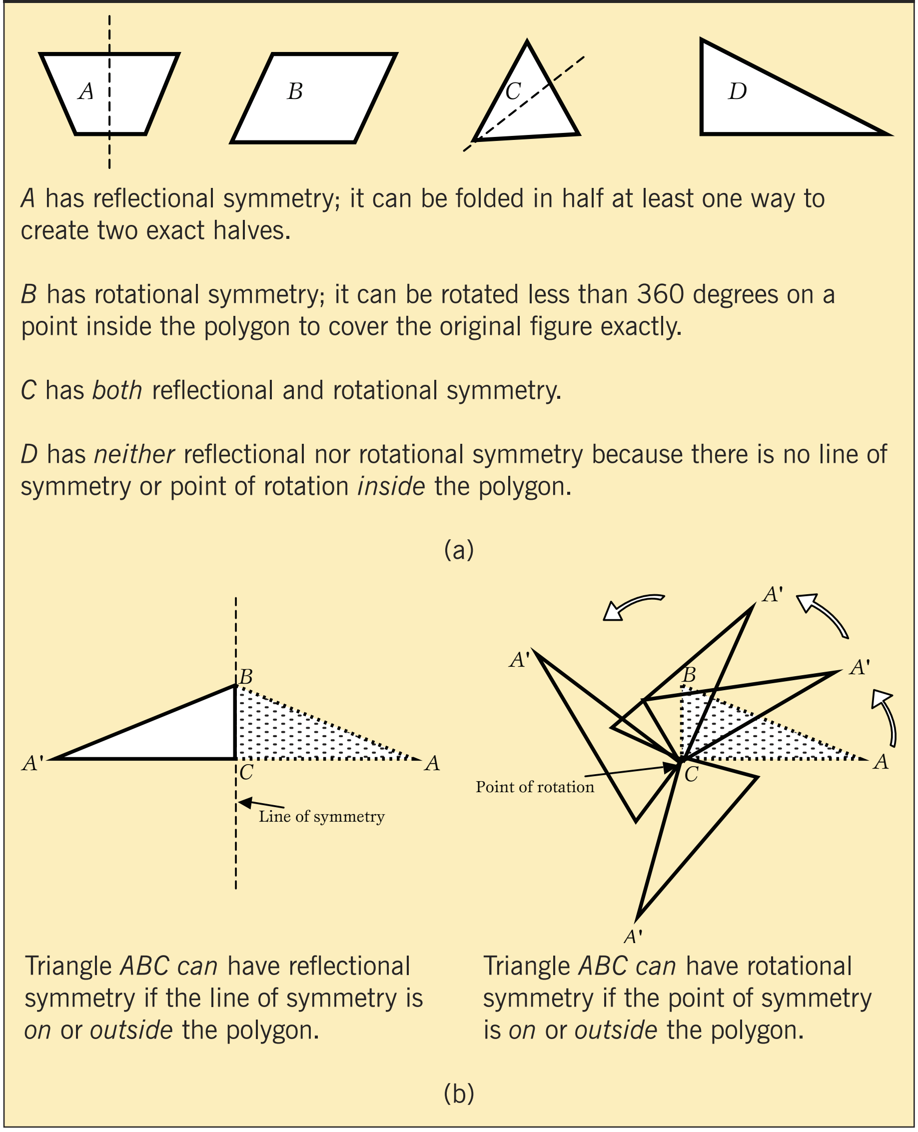
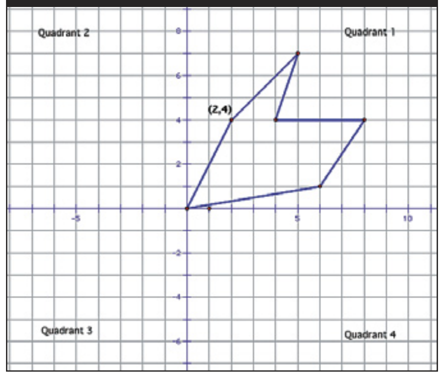


Fig. 2 Students used a polygon in the first quadrant, with a vertex at (0, 0), to begin the investigation of symmetry and rotation.



THE REFLECTION TASK

The first quadrant in the coordinate grid was familiar to the fifth graders, but this was their first exposure to all four quadrants. Student comments included "It's like looking through a scope at a target" and "Don't they have something like that in airplane cockpits?" It was appropriate at this time to briefly discuss negative numbers. We talked about the two axes being number lines that go on forever, extending in the four directions. After we reviewed how to locate points in the first quadrant, the *x*-value first then the *y*-value, we applied the same rule to the other quadrants. Students needed to see several examples before they were comfortable thinking about the coordinate order and whether the values would be negative or positive. When I asked the class to name the location of a particular point and nearly all hands were raised, I knew we were ready to move on.

The Sketchpad feature of *snap points* in the Grid option menu allows points to be plotted on the integer intersections of the grid as the cursor is moved over or near any intersection. Creating a figure becomes easy and very neat, and the labeling of the points is accurate. To model this, I selected my line segment tool to create an original polygon with about seven vertices in the first quadrant and a vertex at (0, 0) (see **fig. 2**). We labeled the points, and again I reinforced the fact that the horizontal *x*-value is always the first number in the coordinate pair.

"Now if I want to reflect my polygon into the second quadrant, what is my line of symmetry?" I asked.

"The *y*-axis," students replied.

"Correct. Let's begin with one of the segments that has an endpoint at (0, 0)," I said.

I selected one line segment of the polygon from (0, 0) to (2, 4) so that it was highlighted, and asked where they thought the endpoint of the segment would be if we reflected it over the *y*-axis. A student walked to the screen and described the new segment on the grid whose endpoints were (0, 0) and (–2, 4). After three or four segments of the original polygon had been reflected by eye and the points labeled, I prompted the students to look at how the point labels were related to their counterparts in the first quadrant. Several students burst forth with the discovery that the first quadrant points (*x*, *y*) translated to (–*x*, *y*) as they were reflected over the *y*-axis (see **fig. 3a**). We continued creating the rest of the segments by renaming the points.

(Note that when students provided answers, I constructed their line segment as described, regardless of whether it was correct. I allowed self-correcting after the student had seen the new segment created. Other students, whose vantage point was farther from the screen, could also suggest a correction.)

We then shifted our focus to the fourth quadrant and began to plot reflected points using the *x*-axis as the line of symmetry. The students, poised to look for a relationship between the coordinate points, soon discovered that the original points (*x*, *y*) translated to (*x*, –*y*) in the fourth quadrant.

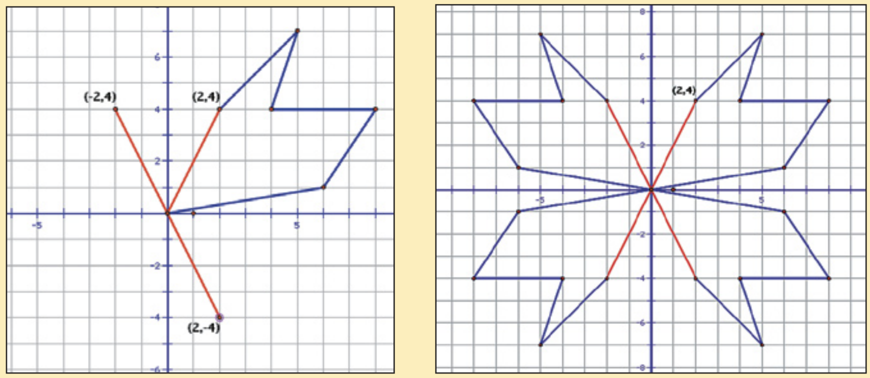
A question arose regarding the third quadrant: "Should it be reflected from the second quadrant or from the fourth quadrant?" "Let's try it both ways and see what happens." After creating the first segment visually from both quadrants, students eagerly concluded that it did not matter if the second or the fourth quadrant image was reflected to obtain the image in the third quadrant: The image will be identical. A quick analysis of the coordinate points revealed that the original first quadrant point (*x*, *y*) translated to the third quadrant coordinate pair (–*x*, –*y*) (see **fig. 3b**).

At this point, the students were itching to get to their own computers. I gave them the parameters of the assignment:

- Create your own polygon in the first quadrant with about 10 points.
- Be sure that both the *x*- and *y*-coordinates are less than 10.
- Label the points.
- Redraw your polygon so that it reflects over both axes and reflects again into the third quadrant.

As they went to their own computers, I touched base with certain students and allowed them the option to reduce or increase the number of points they could use.

Fig. 3 From a polygon in the first quadrant, students found the *x*-axis and *y*-axis reflections of one side of the polygon (a). Then they reflected the image into the third quadrant, and repeated the process for all sides of the polygon and quadrants (b).



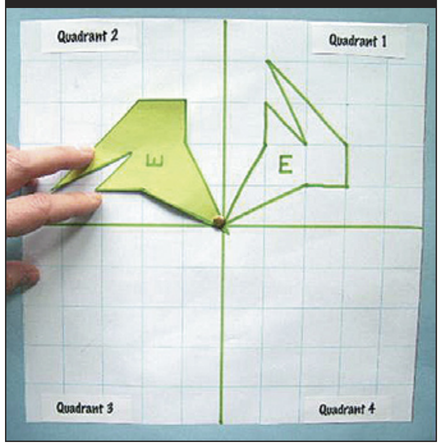
(a) (b)

What fascinated me as I interacted with the students is that some chose to reflect their figure by sight "because it looks right." They later went back to focus on labeling the reflected coordinate points. Other students deferred to the pattern of the coordinate point labels as they plotted their reflected figures. Although students who used the eyeball method tended to complete their reflections quicker, their labeled coordinate points often contained errors.

Labeling the points was not part of the process of creating the reflections. Other students depended on the coordinate points to redraw their polygon reflections. Although they were more apt to have accurately named points, those points may have been plotted in the wrong location.

The benefit of using Sketchpad for this project was obvious at this point: No frustrating, messy erasing occurred. It was easy to fix errors on the computer, and students did not get disheartened by having to make corrections. As students finished and as time allowed, I asked them to switch seats with a neighbor to check the other student's reflections. Because each student's polygon was different, they had to concentrate on their peer's reflections to find errors. Meanwhile, I enjoyed standing behind them and eavesdropping on some excellent math reasoning. I also found I had enough time to spend with those students who needed extra assistance.

Fig. 4 By using a physical model, students were able to observe the rotation of a polygon by 90 degrees.

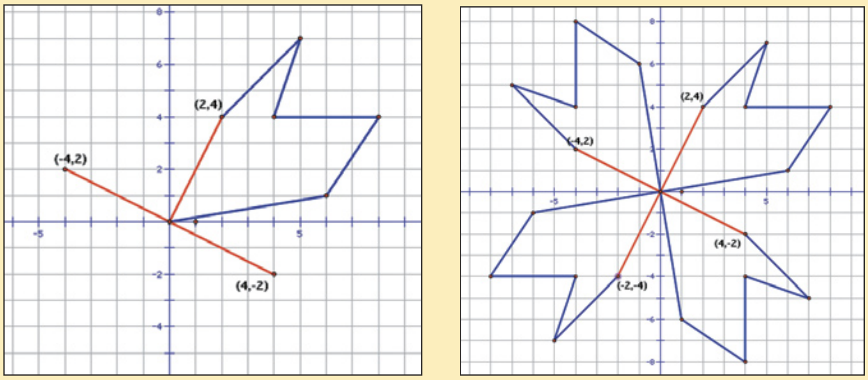


THE ROTATION TASK

The next day we returned to the computer lab to continue the activity. I used a handmade, cardboard model with a brass tack at the point of origin to begin our discussion of rotational symmetry on a coordinate grid (see **fig. 4**).

"If I want to rotate, or turn, this

Fig. 5 From a polygon in the first quadrant, students visualized the 90-degree rotation of one segment of their polygon using Sketchpad (a), then rotated each segment to complete the polygon. Next, they rotated each side of the polygon to obtain the image shown in (b).



(a) (b)

polygon, what point on the grid could I use as my point of rotation?" I asked.

"[The point] (0, 0)."

"Yes. For our purposes we will use (0, 0) as our point of rotation. But we could actually use any point from which to rotate our polygon. If I rotate the polygon all the way around on (0, 0), back to its original position" [I demonstrated the rotation on the model], "how many degrees will it have rotated?"

"Three hundred sixty degrees."

[I am glad that they remembered this fact from studying angles and using a protractor.]

"If I want a rotated copy in each of the four quadrants" [I used the model to exaggerate the rotation by stopping in each of the quadrants], "then how many degrees will it need to rotate each time?"

"Ninety degrees, because 360 degrees ÷ 4 = 90 degrees."

"And what if I rotate it in the other direction?" I asked.

"It's the same thing!"

I modeled this task for students who were not convinced. I then redirected their attention to the screen with my original polygon in the first quadrant from the previous day. I selected the line segment from (0, 0) to (2, 4) so that it was highlighted.

"If I rotate this line segment 90 degrees on point (0, 0) into the second quadrant, where should this segment end?"

Most of the students' initial thinking was that the segment would be in the same position as a reflected segment, at (–2, 4). I drew the new segment from (–2, 4) to the origin, then held up a piece of paper with a corner on (0, 0) and the paper's edge on the original line segment. "Did this line segment rotate 90 degrees?" They could see that it had not. Together we played around with the new line segment until we created a right angle with the original line segment. I mentioned that sometimes it can help to "see" the right angle if they tilted their heads. Finally, the first line segment looked to be rotated correctly (see **fig. 5a**).

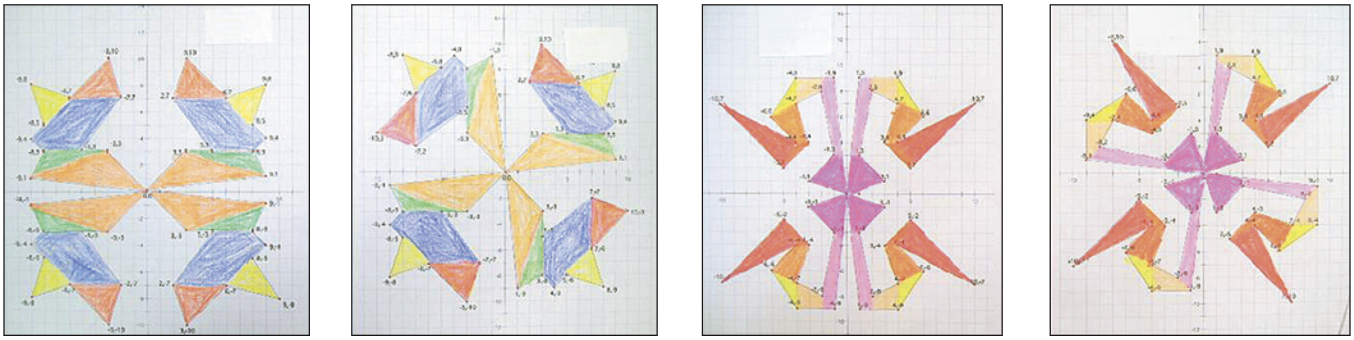
We compared the coordinate points of the two segments from the origin with endpoints (2, 4) and (–4, 2). Someone inevitably noticed that the numbers "switched." When prompted to clarify, the student articulated that the original (x, y) had translated to $(-y, x)$ when rotated. We tested this idea by drawing several more segments of the polygon based on the coordinate point conversion and checking the degree of rotation with the paper's right angle held up to the screen. After drawing several more segments, identifying points by their label rather than by "sight," the rotated image became clearer and easier to confirm by sight as well.

Together we analyzed how the polygon rotated from the first to the fourth quadrant. "We'll need to tilt our heads the other way to see the 90 degrees now," and we all did so to draw the first segment, again, checking with the paper corner. Without much prompting, the students looked for a relationship between the coordinate pairs. "The numbers switched again!" someone called out. We concluded that the original (x, y) translated to $(y, -x)$ (see **fig. 5b**). By modeling the rotation of several segments and asking students to help, I could determine who was comfortable with the new concept. When more hands were up than not, I knew that they were ready to work independently.

Similar to the task from the day before, students used their original polygon and rotated it every 90 degrees, using (0, 0) as the center of rotation. I told them that they needed to discover for themselves what happened to the coordinates in the third quadrant when rotated. On their own, they learned that the third-quadrant coordinates from the original (x, y) rotate to $(-x, -y)$ in the third quadrant.

While walking around the lab as students were working, I observed them using their fingers to help guide them to plot points correctly on the screen, using the edge of their hands against the screen to help them see a rotation, or tilting their heads to check a right angle. I loved seeing these interactions (aside from the smudged computer screens) because they brought a kinesthetic dimension to the use of the geometry software.

I also noticed those visually



Students selected their own polygons and produced a design using reflections in each axis, then used 90 degree rotations.

oriented students who depended on what looks right to create their reflections. They typically had greater difficulty creating their rotation because it was harder to see once they began creating the segments farther away from (0, 0). After some frustration, because making the reflections was so natural for them, these students became convinced that using the mathematical pattern of the coordinate pairs was more practical, although I continued to encourage them to confirm it visually. The students who depended on the coordinate-pair conversions for their reflections generally defaulted to the same method for their rotations right away and were quite successful.

After printing the two symmetries, I asked the students to color each of their figures in the same way; if blue was used in a particular corner, the corresponding corner on all figures should be blue, regardless of its position in the quadrant. This made the rotation and reflection really "pop" visually. An observer could focus on one color and follow the color around in a circle for the rotations and see it flipped, in a similar way to a mirror image for the reflections.

The students enjoyed seeing all the images displayed. I had them focus on the third quadrants of each reflection/rotation pair. They noticed that the third quadrants were exactly the same. They were amazed that even though there was variety in the polygons within each reflection/rotation pair, the third quadrants matched exactly when they were rotated or reflected. When asked why, the students went back to the coordinate pairs and realized that for both the reflection and the rotation, the third quadrants' points were translated to $(-x, -y)$ from the original first quadrant's (x, y) points.

As a final culminating demonstration, I showed my fifth graders how Sketchpad can actually reflect or rotate figures for us. I highlighted my entire original figure, used the reflect tool option, and selected the y-axis as my line of symmetry. My polygon was reflected for me. I demonstrated again, this time reflecting over the x-axis. To show them rotational symmetry, I used my original polygon and invoked the rotate tool. I indicated the degree of rotation, as well as the center of rotation. Compared with the intense concentration that students exhibited with their images the past two days, this seemed ridiculously easy. Their reaction was jaw dropping.

"I did all that hard work for nothing?"

I reminded them how much they learned doing the transformations manually, how the concepts will stick with them, and that they will learn to appreciate the power of technology.

TEACHER'S REFLECTION

I asked myself if I succeeded in my goal of helping students with a spatial deficit find an entry point that afforded them some success in geometry. I think so. This project appeals to a student's ability to see number patterns with the coordinate pairs and uses that strength to create the geometric figures, instead of beginning with their spatial sense and fine-motor drawing. If someone looked at the display wall, the image created by this type of student would not stand out. Sketchpad has eliminated the messiness that can so easily hinder accuracy in geometry. In so doing, the computer has leveled the playing field and equalized the visual value of the product compared with classmates who might be more spatially inclined.

In the past, when I have attempted symmetry or tessellation projects by hand, students' interest would wane as their frustration with lack of accuracy increased. After completing this transformation project using technology, all students were proud of their finished product and eagerly pointed out to friends their symmetry image hanging in the hall.

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