TAPPING INTO TRAPEZOIDS

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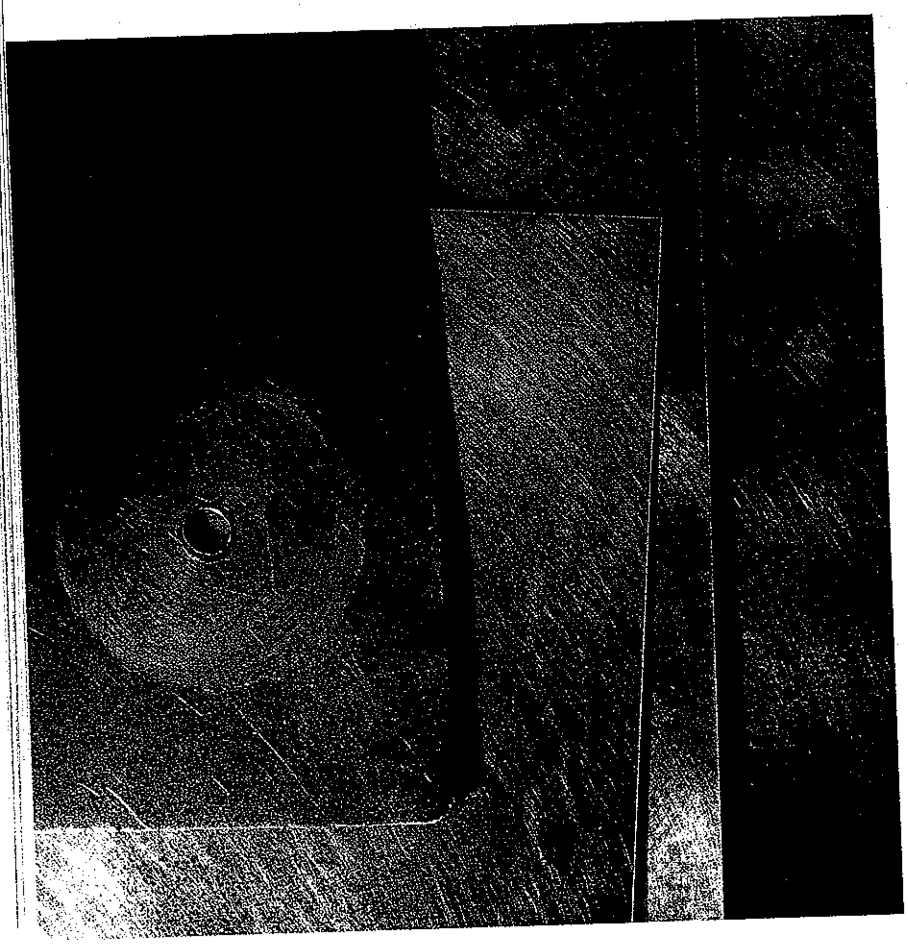


For many students, their only experience with trapezoids centers around the formula $A = (1/2)(b_1 + b_2)h$—memorizing it and applying it in the rare situation that they must calculate the area of a trapezoid. But trapezoids have a great deal of potential in the school mathematics curriculum: for understanding the role of definition in mathematics; for investigating other mathematical ideas, such as symmetry and similarity; for understanding examples of symbolic manipulation and multiple representations; and for looking at the Pythagorean theorem in a different, historical light. In this article, I present these and other ideas in the context of lessons and investigations I have used with preservice middle and high school mathematics teachers in an effort to expand on their understanding of a single geometric shape and to help them discover some of the connections between different mathematical concepts.

WHAT IS A TRAPEZOID?

One of the first problems preservice teachers encounter is establishing a definition of *trapezoid.* Most of the teachers remember a definition they learned in school. I like to have them explore the etymology of *trapezoid* by researching the history of the word at the library and online. I follow this investigation with a discussion about the various ways in which the word *trapezoid* is defined. The following represents some of what we have learned about trapezoids and the different definitions that exist.

The word *trapezoid* comes to us from the Greek word *trapeza*, which means "table" and which is itself made up of *tetra* (meaning "four") and the root *ped-* (meaning "foot"). The suffix *-oid* means "looking like," so a trapezoid is "a figure that looks like a table (at least in somebody's imagination)" (Schwartzman 1994, p. 225).

The trouble with defining a trapezoid from a mathematical perspective, though, is that not everyone agrees upon what the defining properties are. In fact, there are two camps in the United States. First, there are those who define a trapezoid as a quadrilateral with *at least one pair* of parallel sides (**fig. 1**).

This definition is one that is favored by some mathematicians, since it puts all parallelograms (including rectangles and squares) into the set of trapezoids, recognizing that the trapezoid area formula $[A = (1/2)(b_1 + b_2)h]$ could actually be used to find the area of any of the shapes in this set. This definition has already started to appear in some textbooks in the United States, although it seems unlikely that it will become the standard definition.

The more traditional definition—the one known by most preservice teachers—says that a trapezoid is a quadrilateral with *exactly one pair* of parallel sides (**fig. 2**). This definition, which is more widely used in U.S. schools and textbooks, keeps the trapezoid as a shape that is distinctly different from all other quadrilaterals by limiting the number of pairs of parallel sides. We use this definition in the following problems.

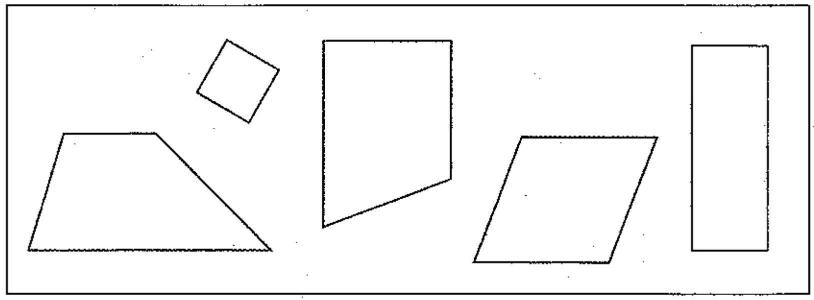


Fig. 1 Trapezoids with at least one pair of parallel sides

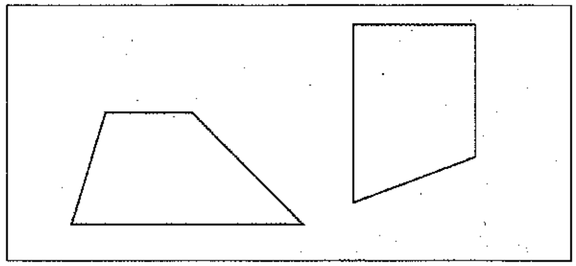


Fig. 2 Trapezoids with exactly one pair of parallel sides

> Make a list of hypotheses about trapezoids that have three congruent sides and a fourth side twice as long as each of the others.

Fig. 3 Special isosceles trapezoid task

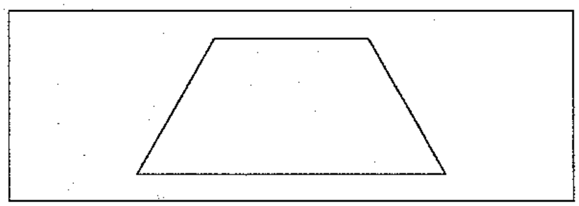


Fig. 4 Drawing of a special isosceles trapezoid

TRAPEZOID PROBLEM 1: THE SPECIAL ISOSCELES TRAPEZOID

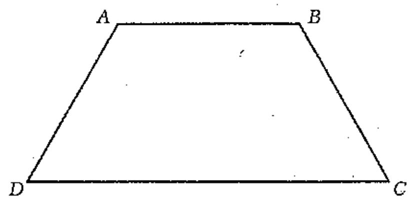
The first trapezoid problem I use with preservice teachers is an example of an open-ended task (**fig. 3**). As such, it allows them to work on the problem at a variety of different levels while raising a number of different conjectures and topics (NCTM 1995). I set no restrictions on the tools they can use to explore this task, and I have observed work with toothpicks (to create side lengths), angle rulers and protractors, and computer software (such as The Geometer's Sketchpad).

The trapezoid used here is referred to as a "special isosceles trapezoid": it is isosceles—the two nonparallel sides have the same length—but more specifically, one of the bases is equal to that length and the other base is twice as long. In this problem, preservice teachers are not given a figure with which to work; instead, they must determine what shape (or shapes) can be made that fits the properties that are given. I have found that there is much to be learned when an exploration is limited to working with a shape of only the given size. One must think about the set of shapes with the given properties and about the fact that they are all mathematically similar. The format of this task as written encourages the participants to develop more general statements and hypotheses.

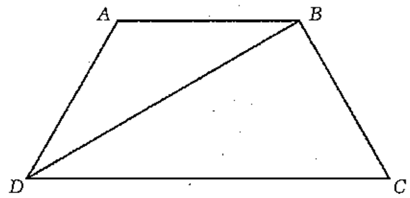
After preservice teachers determine what the set of trapezoids in this task looks like (**fig. 4**), they begin to explore some of the more interesting properties of these trapezoids. They can make hypotheses concerning ideas of congruence, similarity, area, perimeter, angle measures, parallel and perpendicular line segments, and a number of other ideas.

I have often used this task as a part of a two-day experience. I collect a number of the preservice teachers' hypotheses at the end of the first day and then transform some of the hypotheses into questions I want everyone in the class to work on. These questions, which I distribute on the second

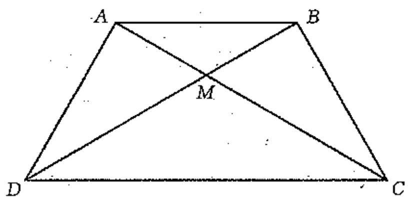
Most people found it helpful to name the vertices of the trapezoid, like those shown here. This helps when referring to vertices and line segments later.



1. By drawing one diagonal as shown, two triangles are created. Accurately describe each of these two triangles using correct mathematical terms. How are their areas related?



2. By drawing both diagonals as shown, four triangles are created. Describe the relationship between $\triangle ADM$ and $\triangle BCM$.



3. Describe the relationship between $\triangle ABM$ and $\triangle CDM$.
4. If perpendicular line segments are drawn from A and B, what are the three shapes that result?

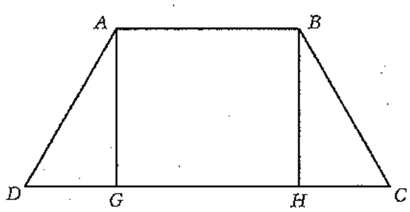


Fig. 5 Sample questions based on preservice teachers' hypotheses

day (see examples in **fig. 5**), provide for a rich class discussion on a number of geometry topics.

For example, question 1 in **figure 5** looks at what happens when one diagonal is drawn in the special isosceles trapezoid. Preservice teachers naturally observe that $\triangle ABD$ is isosceles but are often surprised to find that $\triangle DBC$ appears to be a right triangle. In the following interchange from my classroom, some preservice teachers discuss the nature of proof for a specific triangle and what it takes to show that $\triangle DBC$ is actually a right triangle.

> **Pauline:** I think that $\triangle DBC$ is a right triangle because DC is double BC. I remember something about when a hypotenuse is twice one of the legs . . .
> **Betsy:** But that's to show that you have a 30-60-90 triangle. When you already know that you have a right triangle, you can use that to show that $\angle BCD$ is 60°.
> **Lyle:** So could we use it the other way around? If we know that $\angle BCD$ is 60°, wouldn't that show that $\angle CBD$ is a right angle?
> **Betsy:** I don't think so. What if the smaller angle [$\angle BDC$] is something other than 30°? Then you wouldn't be guaranteed to have a right triangle.
> **Lyle:** But if you know you have a 30° angle and a 60° angle, then you already *know* that the remaining angle is a right angle. You wouldn't even have to use the fact that CD is twice as long as BC.
> **JW:** So what I'm hearing here is a debate about how much information is necessary to show whether $\triangle DBC$ is a right triangle. One idea that is on the floor right now is if it's sufficient to know two side lengths—DC and CB—and the angle between them. In this case, if you know that $\angle BCD$ is 60°, does that determine the triangle? That is, is there only one triangle that you could have?
> **Kyla:** Yeah, that's the SAS theorem.
> **JW:** OK, now can you show that $\angle BCD$ is 60°?

The conversation continued with what it would take to show that $\angle BCD$ is 60°. Several approaches were presented by different preservice teachers. One approach centered around locating the midpoint P of CD, drawing PA and PB, and arguing that since the bases of the trapezoid are parallel, then $ABPD$ is a rhombus, $\triangle BCP$ is equilateral, and $\angle BCD$ is 60°. Another approach also used the midpoint P of CD and the fact that the area of $\triangle DBC$ is double the area of $\triangle ABD$ (the two triangles have the same height but the base of $\triangle DBC$ is twice the length of the base of $\triangle ABD$), arguing that $\triangle ABD \cong \triangle PDB$, so $BP = DA$ and $\triangle BCP$ is equilateral. The exploration of these and other questions helped to reiterate the importance of the axiomatic system and the investigative approach to teaching.

TRAPEZOID PROBLEM 2: REP-TILING WITH TRAPEZOIDS

From their work in classrooms with students, many preservice teachers are familiar with the red trapezoid in the set of pattern blocks. The following task utilizes a set of trapezoid pattern blocks and works well as a follow-up to the previous task, as preservice teachers immediately recognize it as a special isosceles trapezoid.

In this task, the trapezoid pattern block is used in investigating similarity, scale factors, and areas using the concept of *rep-tiles*. Martin Gardner traces the history of rep-tiles (or "replicating figures") back to 1962, when Solomon W. Golomb gave a name to these figures, which can be created from exact mathematically similar replicas of the original figure (Gardner 2001). Rep-tiles also appear as a part of an investigation on similar figures in the Connected Mathematics book *Stretching and Shrinking*, which investigates ideas of similarity (Lappan et al. 2002).

Preservice teachers are given a set of red trapezoid pattern blocks, which they use to create larger trapezoids that are similar to the original—that is,

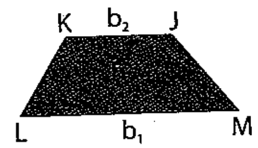
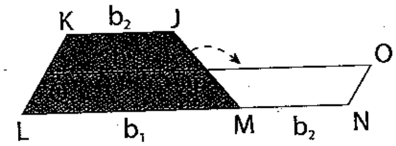
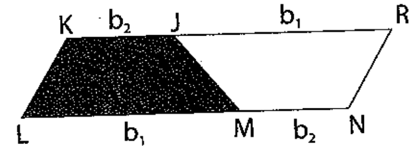
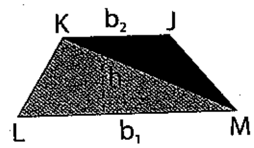
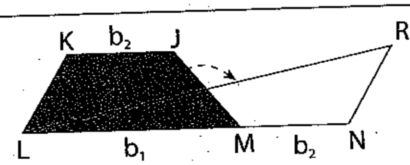
	Mathematical Language	Symbolic	Figure with Manipulation
	(Original trapezoid)		
A			
B			
C			
D			

Fig. 10 "Area of a trapezoid" activity sheet

In 1876, James A. Garfield was serving in the U.S. House of Representatives. He published the following diagram as part of his proof of the Pythagorean theorem. Explain the use of this diagram in the proof.

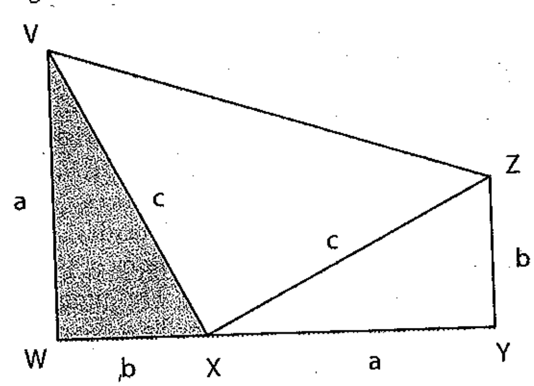


Fig. 11 "Pythagorean theorem" task

orem (**fig. 11**). This puzzle is one of the hundreds of different proofs presented in Elisha S. Loomis's classic book *The Pythagorean Proposition* and is the only one that is attributed to a former U.S. president (Loomis 1940).

In 1876, the editor of the *New England Journal of Education* published this proof, noting that it had been submitted by James A. Garfield, who was then a congressman in the U.S. House of Representatives. Garfield constructed a trapezoid using several triangles and proved the Pythagorean theorem using areas of triangles and trapezoids, binomial expansion, and symbol manipulation (Hill 2002). Garfield noted that he and several other congressmen had hit on it during "some mathematical amusements" and that "we think it [is] something on which the members of both houses can unite without distinction of party" (Gardner 1963, p. 155).

First, I ask the preservice teachers what they recall about proving the Pythagorean theorem. Those who do remember a proof typically recall one, from their high school geometry course, involving similar triangles. I present **figure 11** to preservice teachers as a "Proof without Words" of the Pythagorean theorem and ask them to figure out how this diagram can be used in a proof of the Pythagorean theorem.

Preservice teachers can approach this problem by recognizing that the entire shape *VWYZ* is a trapezoid and writing the area of the trapezoid in terms of *a* and *b*.