

Problem Solving in Middle-Level Geometry

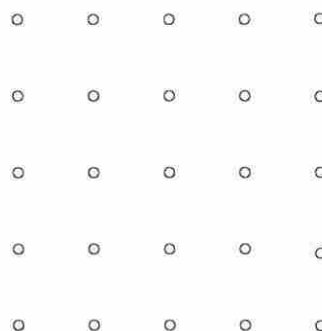
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Few people would disagree that middle school students must engage in legitimate problem-solving experiences in mathematics. Leaders in the world of mathematics education certainly espouse this view (National Council of Teachers of Mathematics 1989, 2000), as do politicians, businesspeople, parents, and concerned citizens. In this article, which is based on the premise that problem solving is vitally important, I offer a tangible suggestion for how dot paper can be used to implement an exploratory approach to problem solving in middle school geometry. Furthermore, I contend that the major instructional ideas put forth in this context can be generalized to other grade levels and other topics within mathematics.

For those unfamiliar with dot paper, it is a remarkably powerful teaching tool that is quite simple in design. Imagine regular graph paper that has small dots at the places where the lines intersect. Now remove all the lines, leaving only the dots. This is known as rectangular dot paper, because dots that are directly vertical or horizontal from one another are equidistant. Woodward and Hamel (1993) suggest that the use of dot paper can expand student thinking and promote active learning in the classroom.

I present two tasks in this article, each of which involves exploration and problem solving. The reader is asked to solve the problems. I will present solutions, a rationale for why a teacher might want to use such tasks with students, and an explanation of how students will benefit from such activities. I should note that the tasks themselves assume very little specific mathematical knowledge, and thus solutions are attainable by any reader, not just teachers of mathematics.

FIGURE 1. A Blank 5 x 5 Grid



Task Number 1

Begin with a 5 x 5 grid on dot paper. Using only vertical and horizontal segments, find all the different ways that you can divide the grid into exactly two polygons of equal area.

Figure 1 shows what the starting grid looks like. Figure 2 shows a rudimentary, but correct, solution. Figure 3 shows a more complex correct solution. Figure 4 shows an example that is *not* a correct answer.

Discussion

At this point in class, I ask the students to play around with the problem. One reason I like this task is that it provides an excellent (and legitimate) opportunity for students to interact with one another. Steele (1993) documents that when students work together on mathematical problems, they express positive views about mathematics, are more likely to be engaged, and

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FIGURE 2. Two Congruent Polygons of Equal Area

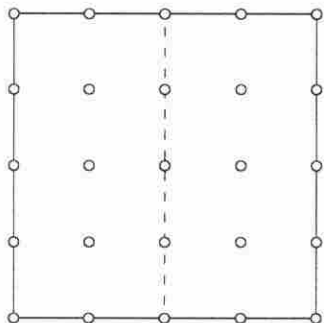


FIGURE 3. Two Incongruent Polygons of Equal Area

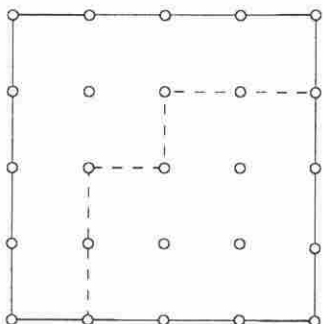
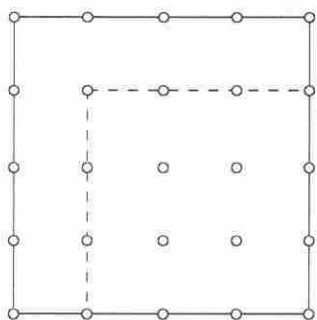
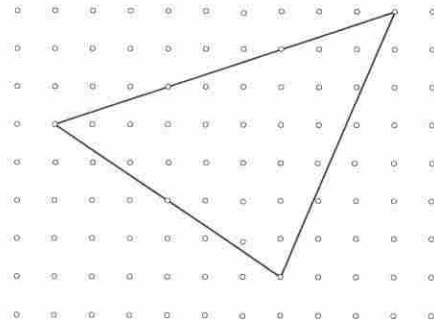


FIGURE 4. Two Polygons without Equal Area



show internal motivation to do well. As students decide if they have found all the solutions, they inevitably compare their answers with one another. Far from being "cheating," this conversation can lead to rich mathematical discussions. One example of a discussion that often surfaces is whether an answer that resembles figure 1 but has the dashed line drawn horizontally through the middle of the grid (rather than vertically) is truly a different answer. This question introduces a "big idea" in both a pedagogical and a

FIGURE 5. Sample Triangle

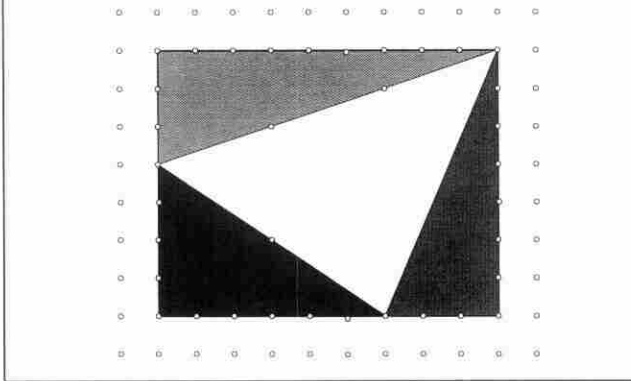


mathematical sense: mathematically, because it paves the way for a discussion about transformational geometry; pedagogically, because it does so within a naturally occurring situation. That is, the idea of transformation enters the realm of our work because it is relevant to that work. Students have created a need to learn about rotations through their own exploration. We often have lively discussions, and the students in the end frequently make the rule that a figure is not a new solution if it can be obtained by simply rotating an existing solution. That is a significant idea, and the critical point is that the students generated the idea, not the teacher.

Another appealing aspect of this problem from a general instructional viewpoint is that it is open-ended in the sense of having multiple correct solutions. Carroll (1998) notes that open-ended questions encourage creative student thinking. In addition, having multiple solutions creates the curious phenomenon that a problem can simultaneously be accessible to every student in the class yet still be challenging to the most experienced students. The student who typically struggles with mathematics will be able to succeed at one level, while the student who excels in mathematics may find it difficult to determine all the solutions. Even the very idea of finding "all" the solutions can lead to an interesting mathematical question: "How do we know when we have found all the possible arrangements?" Such questions give students a sense of what it truly means to do mathematics. You may be wondering how many different arrangements exist; let's just say there are more than ten.

I realize this is only a single problem, and there is a limit to how valuable any one problem can be. However, the point to make is that we must consider the *type* of problem we use. I urge teachers to think broadly and creatively when designing tasks, so that students learn as much as possible from their engagement with those tasks. As English (1998) demonstrated, tasks that focus on important reasoning processes, rather than

FIGURE 6. First Solution to Finding Area of Central Triangle



computation, have the greatest potential for student engagement.

Task Number 2

Find the area of the triangle drawn on the dot paper in figure 5. You may use any method you choose. As a reminder:

$$\text{Area of rectangle} = (\text{length})(\text{width})$$

$$\text{Area of triangle} = (1/2)(\text{base})(\text{height})$$

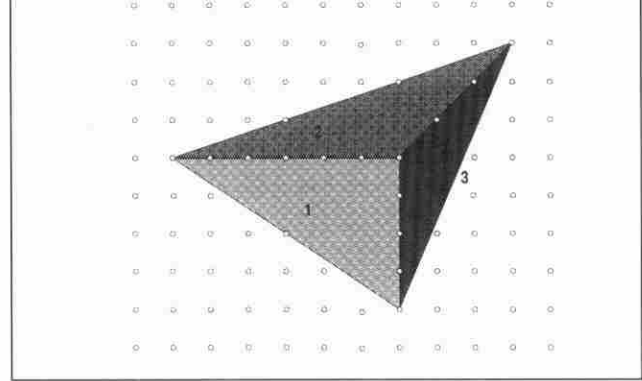
Finally—and this is the real challenge—the goal is to find three unique methods for solving this problem.

Discussion

Students often begin this task by trying to plug numbers into the area formula for triangles. They soon realize, however, that this approach is limited. The original triangle is drawn so that there are no convenient numbers to use for the base and the height. Indeed, if such numbers were readily available, this task would not be a problem at all, but rather a routine exercise.

Thus we come to another significant instructional idea, one that can transfer to many situations. As teachers, we help students who are faced with a new challenge by building on what they already know. Many researchers, including Steffe and D'Ambrosio (1995), support the idea of activating students' prior knowledge to make sense of a new situation. In this case, what students know is how to calculate the area of rectangles and of triangles where the base and height are obvious. When reminded of this, students often produce a solution that is diagrammed in figure 6. A rectangular box is drawn around the original triangle. The area of this rectangle is easy to compute (either by a formula, or by counting the number of "grid squares" that are contained within it). The shading in the figure shows that this new rectangle can be viewed as a collection of triangles: the original one and the three new, shaded triangles. Notice that the three shaded triangles are all right triangles, which makes their area easy to

FIGURE 7. Second Solution to Finding Area of Central Triangle



determine. The solution method tends to become apparent at this point: the area of the desired triangle can be determined by subtracting the areas of the three shaded triangles from the area of the rectangle. The rectangle is 63 square units; the shaded triangles are 12 square units, 13 and 1/2 square units, and 10 and 1/2 square units. The shaded triangles total 36, which we subtract from 63, leaving 27 square units as the area of the original triangle.

We have produced one solution—not trivial, yet accessible to any middle school student who has worked with triangles, rectangles, and simple arithmetic involving fractions. A seemingly straightforward area task now becomes challenging, however, as we must produce two more unique approaches that will lead to a correct answer.

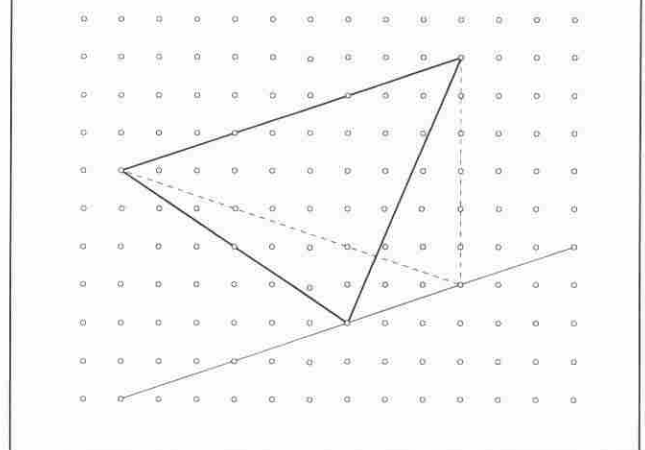
Figure 7 shows an approach that might be inspired by the first solution. Some students may think of this approach first. Rather than "boxing" around the triangle, the student subdivides the original triangle into smaller pieces, then adds up the areas of the small pieces to find the area of the whole. In some ways, this approach may be more intuitive than the first solution in that it relies on a fundamental idea of area that is developed at a young age: "area" can be thought of as "covering," and covering can be accomplished with numerous arrangements of pieces without changing the total area (Van de Walle 1998). What makes this approach complicated is choosing how to subdivide the original triangle so that the area of the smaller pieces can be computed easily. The solution in figure 7 shows one of many ways to subdivide the original triangle. Alternate subdivisions may include squares, trapezoids, and other polygons. Teachers may want to discuss with students whether partitioning the triangle into different arrangements truly constitutes a "new solution" or whether all such partitions are really just variations on the broader idea of subdivision. Asking students to participate in this type of discussion is part

of the teacher's broader task (National Council of Teachers of Mathematics 1991) of creating a classroom environment that welcomes mathematical discourse and values student input. The students share the responsibility for deciding the validity of a solution within the structure of a discipline. In this case the discipline is mathematics, but the strategy of asking students to validate solutions applies to other disciplines.

I will elaborate on the solution in figure 7 because the work demonstrates an important pedagogical idea. A teacher, through carefully chosen tasks, can present mathematical concepts within the context of interesting problems, rather than in isolation. In this case, we first look at small triangle 1, whose area can be determined directly because it is a right triangle. However, triangles 2 and 3 pose more substantive challenges. At first glance, many students will find it difficult to compute the area because they will not see any convenient height and base to use in the area formula. A teacher who has worked with students on this type of area task has likely encountered a common misconception: some students erroneously think that an altitude (the technical term for height) must occur "inside" the triangle. We have all sought ways to help students overcome this misconception, with varying degrees of success. I posit that in this task, the combination of embedding the idea of altitudes within a challenging problem and using dot paper as a visual tool makes it meaningful and relevant for students to look outside the triangle to find an altitude. In small triangle 2, for instance, if we take the horizontal line as the base of length 6, then the altitude can be drawn from the upper right vertex perpendicularly to that base extended, for a height of 3. Once this base/height combination is determined, finding the area is routine. A similar process is applied to small triangle 3. When we generate the sum of the areas of all three small triangles, we find it equals 27, which is exactly the answer we determined in the first solution.

We now have two unique solutions. Can you think of a third way to solve this problem? I have seen a variety of methods, some of which are variations on the first two solutions, some of which are both original and highly creative. My current favorite is demonstrated in figure 8. I would not necessarily expect a middle school student to generate this particular solution on his or her own. However, I include this solution for three reasons. First, the concepts involved are within the cognitive grasp of middle school students. Second, dot paper clarifies the mathematics particularly well. Third, the solution introduces a powerful mathematical idea that is often overlooked, does not need to be overlooked, and turns out to be at the heart of numerous proofs of the Pythagorean theorem. Although middle school may not be the place to ask students to produce rigorous proofs of the Pythagorean theorem, it is

FIGURE 8. Third Solution to Finding Area of Central Triangle



a great place to begin learning the foundational ideas that later form the structure of mathematical proofs. Thus, the pedagogical rationale behind this solution is to use a seemingly straightforward task as an avenue for introducing a robust mathematical concept.

Now to the solution: figure 8 shows the original triangle in bold, a new dashed triangle that shares one side with the original triangle, and an auxiliary line. The auxiliary line is drawn intentionally to be parallel to the side the two triangles share in common. I usually pose the following questions to students: "Any thoughts on why we want the two lines to be parallel?" "How can we use the dot paper to convince someone that the lines are indeed parallel?" "I know we haven't studied this directly, but does anyone want to make a conjecture about the relationship between the two triangles?" Students are asked to describe the thinking behind any responses they give, and depending on the ideas elicited, a discussion may ensue. You may want to ponder the questions before reading further.

If we think of the two triangles as having two fixed points in common, we now introduce the innovative idea that the dashed triangle can be obtained by simply sliding the third vertex of the bold triangle along the auxiliary line until it rests in its position as the third vertex of the dashed triangle. After letting students think about this for a bit, I again pose the same questions listed above. Building on what students in middle school already know, we recall that parallel lines are always the same distance apart. Thus, each vertex that lies on the auxiliary line is an equal distance from the common side of the two triangles. Think of this distance as an altitude and the common side as the base, and we can conclude that the two triangles have the same area as each other. Believe it? It's true, from the straightforward reasoning that if the bases are equal and the altitudes are equal, then the areas must be

equal. This phenomenon of dragging a vertex along a parallel line, thereby changing the shape but not the area, forms the core of the ingenious mathematical idea known as "shearing." Students are amazed when they consider that this same line of reasoning can be used to show that the area of the bold triangle will stay the same no matter where we slide the vertex along the auxiliary line. As a side note, dynamic geometry software, such as Geometer's Sketchpad or Cabri, offers a very convincing visual demonstration of this.

Some of you may be thinking, That's great, but we still don't know the area of either triangle. True, but through shearing we have created the dashed triangle, whose area we can compute without great difficulty. Think of the vertical side of the dashed triangle as its base, of length 6. The height is now simply counted from this base to the opposite vertex, giving length 9. We invoke the area formula, yielding the same area 27 that we found in the other two solutions.

Conclusion

In this article I intended, first, to present some intriguing mathematical tasks that would demonstrate the elegance of mathematics, the usefulness of dot paper as a teaching tool, and the value of divergent thinking approaches to problem solving. My second, and more formidable, goal was to use the activities in the paper as a springboard for a larger discussion of instructional strategies that motivate students and enhance understanding.

In particular, tasks such as the ones in this article allow students to engage in meaningful discussion and thereby learn from one another. Students generate their own ideas, the hallmark of true learning and thinking, rather than simply repeating ideas that are given to them. The tasks themselves are challenging

and nonroutine yet are based on fundamental, familiar principles. In this way, students apply what they already know to a new situation and extend their knowledge base. The problems allow for creativity and reward multiple approaches. It is quite likely that some students will generate solutions that are new to the teacher, a feat that should be celebrated. What better mark of learning than to have a student present a solution that we haven't seen before? Finally, by carefully choosing rich problems, a teacher creates an environment where every student has a chance to work at his or her level, and all students gain insight into what it means to engage in mathematical problem solving.

Key words: mathematics, middle school, geometry, dot paper, problem solving

REFERENCES

- Carroll, William. 1998. Middle school students' reasoning about geometric situations. *Mathematics Teaching in the Middle School* 3 (6): 398-403.
- English, Lyn D. 1998. Children's perspective on the engagement potential of mathematical problem tasks. *School Science and Mathematics* 98 (2): 67-75.
- National Council of Teachers of Mathematics. 2000. *Principles and standards of school mathematics*. Reston, VA: Author.
- . 1991. *Professional standards for teaching mathematics*. Reston, VA: Author.
- . 1989. *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- Steele, Diana F. 1993. What mathematics students can teach us about educational engagement: Lessons from the middle school. ERIC, ED 370768.
- Steffe, L., and B. D'Ambrosio. 1995. Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education* 26 (2): 146-59.
- Van de Walle, John A. 1998. *Elementary and middle school mathematics: Teaching developmentally*. New York: Longman.
- Woodward, E., and T. R. Hamel. 1993. The use of dot paper in geometry lessons. *The Mathematics Teacher* 86 (7): 558-61.