Chapter 7  Modeling Periodic Behavior

Section 7.1  Introduction to the Sine and Cosine Functions

1. The length of Janis’s fingernails is a periodic function with a period of 1 week.

2. Since the length of months varies, this function is not perfectly periodic. However, the pattern of months repeats every four years, so as long as Harry does not change the style of his haircut, or go (prematurely) bald, the length of his hair will be periodic with period 4 years.

3. The figure illustrates the graphing process for angles between $0^\circ$ and $90^\circ$.

In Problems 4 and 5, to compare the measure of an angle $\theta$ in degrees to its measure in radians we use the proportion $\frac{\theta}{\pi} = \frac{\theta^\circ}{180^\circ}$.

4. a. $15^\circ$ corresponds to $\pi/12$ radians  
   b. $75^\circ$ corresponds to $5\pi/12$ radians  
   c. $120^\circ$ corresponds to $2\pi/3$ radians  
   d. $150^\circ$ corresponds to $5\pi/6$ radians  
   e. $225^\circ$ corresponds to $5\pi/4$ radians  
   f. $315^\circ$ corresponds to $7\pi/4$ radians  
   g. $270^\circ$ corresponds to $3\pi/2$ radians  
   h. $240^\circ$ corresponds to $4\pi/3$ radians  
   i. $-135^\circ$ corresponds to $-3\pi/4$ radians  
   j. $-210^\circ$ corresponds to $-7\pi/6$ radians

5. a. $3\pi/4$ radians corresponds to $135^\circ$  
   b. $4\pi/5$ radians corresponds to $144^\circ$  
   c. $2\pi/3$ radians corresponds to $120^\circ$  
   d. $1.5$ radians corresponds to $86^\circ$  
   e. $2.5$ radians corresponds to $143^\circ$  
   f. $3$ radians corresponds to $172^\circ$  
   g. $\pi/8$ radians corresponds to $22.5^\circ$  
   h. $5\pi/3$ radians corresponds to $300^\circ$  
   i. $-3\pi/2$ radians corresponds to $-270^\circ$  
   j. $-5\pi/3$ radians corresponds to $-300^\circ$
6.  
   a.  \( f(30') = \frac{5}{2} \)
   b.  \( f(45') = 5 \frac{\sqrt{2}}{2} \)
   c.  \( f(60') = 5 \frac{\sqrt{3}}{2} \)
   d.  \( f(120') = 5 \frac{\sqrt{3}}{2} \)
   e.  \( f(-15') = -1.294095 \)
   f.  \( f(873') = 2.26995 \)
   g.  \( f(\pi/4) = 5 \frac{\sqrt{2}}{2} \)
   h.  \( f(\pi/3) = 5 \frac{\sqrt{3}}{2} \)
   i.  \( f(\pi/12) = 1.294095 \)
   j.  \( f(-\pi/6) = -\frac{5}{2} \)
   k.  \( f(5.27) = -4.24261 \)
   l.  \( f(-25.614) = -2.31448 \)

7.  
   a.  \( f(30') = \frac{\sqrt{3}}{2} \)
   b.  \( f(45') = 1 \)
   c.  \( f(120') = -\frac{\sqrt{3}}{2} \)
   d.  \( f(225') = 1 \)
   e.  \( f(\pi/3) = \frac{\sqrt{3}}{2} \)
   f.  \( f(\pi/12) = \frac{1}{2} \)
   g.  \( f(3\pi/8) = \frac{\sqrt{2}}{2} \)
   h.  \( f(2\pi/7) = 0.9749 \)

8.  The cosine function is decreasing for \( \theta \) from 0 to \( \pi \), and increasing for \( \theta \) from \( \pi \) to \( 2\pi \). The cosine function is concave down on the interval from 0 to \( \pi/2 \) and again on the interval \( 3\pi/2 \) to \( 2\pi \). It is concave up on the interval from \( \pi/2 \) to \( 3\pi/2 \). Since cosine is periodic of period \( 2\pi \), these patterns are repeated thereafter. The inflection points of cosine are coincident with its zeros; they are at \( \theta = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \ldots \). The maximum value that the cosine achieves is 1. The cosine function achieves this maximum value at \( \theta = 0, \pm2\pi, \pm4\pi, \ldots \). The cosine function achieves its minimum value of -1 at \( \theta = \pm\pi, \pm3\pi, \pm5\pi, \ldots \).

9.  

The two graphs are the same, since for any \( x \), \( \sin x = \cos \left(x - \frac{\pi}{2}\right) \); the sine graph is the same as the cosine graph shifted \( \pi/2 \) units to the right.

10.  

The two graphs are the same, since for any \( x \), \( \cos x = \sin \left(x + \frac{\pi}{2}\right) \); the cosine graph is the same as the sine graph shifted \( \pi/2 \) units to the left.

11.  a.  The period for the population of lynxes is roughly 10 years.
   b.  The period for the population of hares is also roughly 10 years.
   c.  The lynx population reached its maximum values in the years: 1847, 1857, 1867, 1877, 1886, 1895, 1906, 1915, 1927, and 1936. The lynx population was at a minimum in the years: 1852, 1862, 1872, 1881, 1891, 1901, 1908, 1920, and 1930.
   d.  The hare population reaches its maximum values in 1853, 1857, 1861, 1864, 1873, 1876, 1886, 1896, 1904, 1913, 1923, and 1933. The hare population is at a minimum in 1847, 1855, 1859, 1882, 1899, 1908, 1918 and, 1928.
e. The years in which the population achieves a maximum (minimum) is separated by ten years from the next maximum (minimum).

f. The years in which the hare population passed an inflection point are roughly halfway between a maximum year and the following minimum year or halfway between a minimum year and the following maximum year.

12. Recalling that the graph of \( y = f(x + a) \) is simply the graph of \( y = f(x) \) shifted \( a \) units to the left, we see that the graph of \( \cos x \) is obtained by shifting the graph of \( \sin x \) to the left by \( \pi/2 \) units. That is, \( \cos x = \sin(x + \pi/2) \). Since \( \sin x \) repeats every \( 2\pi \), there are infinitely many possible answers: \( \cos x = \sin(x + \pi/2 + 2n\pi) \) for any integer \( n \geq 0 \).

Section 7.2 Modeling Periodic Behavior with the Sine and Cosine Functions

1. a. The graph depicts a periodic function with period 2.
b. The graph depicts a periodic function with period 2.
c. The graph depicts a periodic function with period 2.
d. The graph depicts a periodic function with period 2.
e. The graph depicts a periodic function with period 2.
f. The graph is not periodic.
g. The graph is not periodic.
h. The graph depicts a periodic function of period 2. Constant functions can be considered periodic, but a definite period cannot be determined.
i. The graph depicts a periodic function with period 2.
j. The graph depicts a periodic function with period 20.

2. The number of hours of daylight in San Diego is well approximated by the function
   \[
   S(t) = 12 + 2.4 \sin \left( \frac{2\pi}{365} (t - 80) \right),
   \]
   where \( t \) is the number of days after December 31. According to the calendar, March 1 corresponds to the value \( t = 60 \), May 12 corresponds to \( t = 132 \), and July 4 corresponds to \( t = 185 \). Hence, \( S(60) = 11.2 \), \( S(132) = 13.9 \), and \( S(185) = 14.3 \).

3. a. The amplitude of \( H \) is 3.6 hours.
b. The period of \( H \) is 365 days.
c. The length of the shortest day of the year is \( 12 - 3.6 = 8.4 \) hours.
d. The length of the longest day is \( 12 + 3.6 = 15.6 \) hours.

4. The equation for the number of hours of daylight for any geographical location south of the Arctic Circle and north of the Antarctic Circle is of the form \( H(t) = 12 + A \sin \left( \frac{2\pi}{365} (t - 80) \right) \). Thus, we need only determine the parameter \( A \), (or the amplitude) for Fairbanks. Since the shortest day in Fairbanks is 3.7 hours, we know that \( 3.7 = 12 - A \), which gives \( A = 8.3 \). So the equation is
   \[
   H(t) = 12 + 8.3 \sin \left( \frac{2\pi}{365} (t - 80) \right).
   \]

5. From the text we know that the number of hours of daylight (from sunrise to sunset) in San Diego is given by \( S(t) = 12 + 2.4 \sin \left( \frac{2\pi}{365} (t - 80) \right) \). Provided we are prepared to interpret “dark” as including twilight (from sunset to sunrise), the required formula is
   \[
   H(t) = 24 - S(t) = 12 - 2.4 \sin \left( \frac{2\pi}{365} (t - 80) \right).
   \]
6. The adjustments to the formula \( y = 9 + 2 \sin \left( \frac{\pi}{6} (t - 7) \right) \) that was found in Example 1 are in the period and the phase shift. The time between high and low tides is 6.5 hours, so a complete tide cycle is 13 hours. Thus the new period is 13. Halfway between 4 AM and 10:30 AM is 7:15 AM, which gives the new phase shift. The new equation for the tide height is \( y = 9 + 2 \sin \left( \frac{2\pi}{13} (t - 7.15) \right) \).

7. Tides are reasonably modeled by formulas of the form \( D + A \sin (B(x - C)) \). For the Bay of Fundy we are given information to conclude that the average tide level is \( D = 25 \), the amplitude is \( A = 25 \), and the frequency is \( B = \frac{2\pi}{11} \). We are not given times for the low tide and high tide, so we cannot determine a phase shift value. Thus any model of the form \( 25 + 25 \sin \left( \frac{(2\pi/11)(x - C)}{1} \right) \) would be appropriate. For simplicity, we could say \( C = 0 \) and get \( W(t) = 25 + 25 \sin \left( \frac{2\pi}{11} t \right) \).

8. a. Modeling the heating of Sylvia’s house with a sinusoidal function yields \( 66 + 2 \sin(2\pi t) \).
   b. Since Gary’s house is further North, we might expect it to cool faster, causing the furnace to cycle faster. This would increase the magnitude of the coefficient of \( t \) in the argument of the sine function.
   c. We might expect the furnace in Jodi’s house to cycle more slowly. This would decrease the magnitude of the coefficient of \( t \) in the argument of the sine function.
   d. From our earlier work, we would expect the cooling part of the cycle to be governed by Newton’s Law of Cooling. This implies that the cooling part of the cycle would be modeled by a decreasing exponential function and be concave up. The downward portion of a sinusoidal model is concave down over its first half.

9. a. A sinusoidal model of an ocean wave will be of the form \( A \sin(Bt) \). If the height of the wave is 4 feet, the amplitude of the model must be 2. Using our rule of thumb, the length of a four-foot wave would be 80 feet, making the period of our model 80. Using these observations, the model becomes \( 2 \sin \left( \frac{\pi}{40} t \right) \).
   b. Reasoning as in part (a), the sinusoidal model will have an amplitude of 7.5 and a period of 300, so the model is \( 7.5 \sin \left( \frac{\pi}{150} t \right) \).

10. a. The period of this sinusoid model is \( \frac{60}{72} = \frac{5}{6} \) seconds per beat.
   b. The frequency of this sinusoid model is \( \frac{72}{60} = \frac{6}{5} = 1.2 \) times per second.
   c. The sinusoidal model is in the form \( D = A \sin Bt \). Since halfway between 80 and 120 is 100, we get 100 for the vertical shift and 20 for the amplitude. The model’s equation is \( 100 + 20 \sin \left( \frac{12\pi}{5} t \right) \).

11. a. Once the oven has reached the target temperature, a function such as \( T = 350 + 10 \sin \left( \frac{2\pi}{10} t \right) \) might be an appropriate model, where \( T \) is temperature and \( t \) is time in minutes, and assume that “cycle” means full cycle from on to off to on again.
   b. If we put the turkey in the oven at 350°, its average temperature will rise to equal the average temperature of the oven and then cycle (at some lag from the oven temperature). If the oven temperature were constant at 350°, Newton’s law of heating would give the turkey temperature as \( T_{\text{turkey}} = 350 - 310e^{-rt/175} \). Since the oven is turning on and off, the temperature will rise faster and some times than others, and once the average temperature has reached 350°, the turkey temperature will cycle through a much narrower range than the oven temperature. If it takes 60 minutes for the turkey to reach 130°, the turkey’s time constant is so large (that is, the turkey responds so slowly to changes in the surrounding temperature) that the temperature fluctuations in the turkey will be less than 0.1°. In fact we’ll take the turkey out of the oven long before it reaches the minimum temperature of 340°, which would happen in about 10 hours.
12. In graph (b) the amplitude changes but the underlying frequency appears to be constant, so this represents an AM signal. Graph (a) shows changing frequency, so this could be an FM signal.

13. a. The period is $2(20 - 8) = 24$ and the amplitude is $\frac{72 - 30}{2} = 21$. The vertical shift is the average of the min and max, or $\frac{30 + 72}{2} = 51$. A sine function with these parameters is $51 + 21\sin\left(\frac{2\pi}{24}(x - C)\right)$. To find a possible value for $C$, note that if $C$ were 0 the first maximum would occur one quarter of the way through the period, at $x = 6$. We want this peak to be at 8 instead, so $C = 2$ and the final formula is $f(x) = 51 + 21\sin\left(\frac{2\pi}{24}(x - 2)\right)$.

b. The cosine parameters will be the same except for $C$. With $C = 0$ the first peak is at $x = 0$, so we need to shift this peak 8 units to the right. Thus $C = 8$ and the final formula is $f(x) = 51 + 21\cos\left(\frac{2\pi}{24}(x - 8)\right)$.

14. a. The period is twice the distance between the inflection points, or $2(18 - 6) = 24$, and the amplitude is $43 - 20 = 43$. Since the inflection points have $y$-coordinate 20, the vertical shift is 20, so a sine function might be $20 + 23\sin\left(\frac{2\pi}{24}(x - C)\right)$. Sine normally has an inflection point at $x = 0$ and we want to move this to $x = 6$, so we can choose $C = 6$. Our final function is $f(x) = 20 + 23\sin\left(\frac{2\pi}{24}(x - 6)\right)$. Note that there are other possibilities: $f(x) = 20 + 23\sin\left(\frac{2\pi}{24}(x + 6)\right)$ would also meet the requirements.

b. Only the phase shift needs to be adjusted for the cosine function. Since the unshifted cosine has an inflection one quarter of the way through the period, we can use $C = 0$ and get $20 + 23\sin\left(\frac{\pi}{12}x\right)$.

15. Assuming that the mean temperature is the average of the high and the low we have a mean of $\frac{64 + (-20)}{2} = 22$ and an amplitude of $\frac{64 - (-20)}{2} = 42$. The period is roughly 365 days, and if we use a sine function we must shift the minimum to the right by one quarter of a period plus 40 more days, that is, by $\frac{365}{4} + 40 = 131.25$ days. A good approximation would be $T(t) = 22 + 42\sin\left(\frac{2\pi}{365}(t - 131)\right)$.

16. a. 

\begin{center}
\begin{tikzpicture}
  \begin{axis}[
    axis lines=middle,
    samples=100,
    domain=0:8*pi,
    xtick={0,\\pi,2*\\pi,3*\\pi,4*\\pi},
    ytick={-3,-1,1,3},
    grid=both,
  ]
    \addplot[color=red,thick,samples=100,domain=0:8*pi]{sin(x)};
    \addplot[color=blue,thick,samples=100,domain=0:8*pi]{cos(x)};
  \end{axis}
\end{tikzpicture}
\end{center}

c. 

\begin{center}
\begin{tikzpicture}
  \begin{axis}[
    axis lines=middle,
    samples=100,
    domain=0:8*pi,
    xtick={0,\\pi,2*\\pi,3*\\pi,4*\\pi},
    ytick={-2,-1,1,2},
    grid=both,
  ]
    \addplot[color=red,thick,samples=100,domain=0:8*pi]{2*sin(x)};
    \addplot[color=blue,thick,samples=100,domain=0:8*pi]{2*cos(x)};
  \end{axis}
\end{tikzpicture}
\end{center}

d. 

\begin{center}
\begin{tikzpicture}
  \begin{axis}[
    axis lines=middle,
    samples=100,
    domain=0:8*pi,
    xtick={0,\\pi,2*\\pi,3*\\pi,4*\\pi},
    ytick={-4,-2,2,4},
    grid=both,
  ]
    \addplot[color=red,thick,samples=100,domain=0:8*pi]{4*sin(x)};
    \addplot[color=blue,thick,samples=100,domain=0:8*pi]{4*cos(x)};
  \end{axis}
\end{tikzpicture}
\end{center}
17. Since trigonometric functions are periodic there are infinitely many correct formulas for each of the graphs shown in the text. For each graph there are two infinite families of formulas, one family based on the sine function, one on the cosine function. Within each family the functions can be seen to differ only in their phase shifts.

a. \( y = 2 \sin \left( \frac{1}{3} x \right) \)  
b. \( y = 5 \sin 2x \)  
c. \( y = 4 \cos 4x \)

d. \( y = -3 \sin \left( \frac{1}{5} x \right) \)  
e. \( y = 1.5 \cos \left( \frac{1}{4} x \right) \)  
f. \( y = 2 - \cos 2x \)

g. \( y = -5 \cos 2x \)  
h. \( y = 10 + 6 \sin 8x \)  
i. \( y = -3 + 3 \sin 4x \)

j. \( y = 3 + 3 \cos 4x \)  
k. \( y = \sin \left( \frac{4}{2} x \right) \)  
l. \( y = 4 \cos \pi x \)

18. A preliminary observation is in order. Note that \( f_1, f_3, f_4, \) and \( f_6 \), are antisymmetric about the origin. This suggests using a formula of the form \( A \sin Bx \) for these functions. For comparison we look at the table of values of \( \sin n \) for \( n = -6, -5, -4, \ldots, 6 \). They are: \( \{0.279, 0.959, 0.757, -0.141, -0.909, -0.841, 0, 0.841, 0.909, 0.141, -0.757, -0.959, -0.279\} \). We immediately identify \( f_1(x) = \sin x \) and incidentally, \( f_2(x) = 2 + \sin x \). A little more inspection reveals that \( f_3(x) = 2 \sin x \). Noticing that the entries in the first and sixth columns add to one gives us \( f_6(x) = 1 - \sin x \). The key to \( f_4 \) is the observation that \( f_4(1) = f_1(2), f_4(2) = f_1(4), \) and so on, which gives us \( f_4(x) = \sin 2x \). Similarly, we see that \( f_5(2) = f_1(1), f_5(4) = f_1(2), \) and so on, suggesting (correctly) that \( f_5(2) = \sin (x/2) \). In summary: \( f_1(x) = \sin x, f_2(x) = 2 + \sin x, f_3(x) = 2 \sin x, f_4(x) = \sin 2x, f_5(x) = \sin (x/2), \) and \( f_6(x) = 1 - \sin x \).

19. The vertical shift is 59. The amplitude of a sinusoidal model would be one-half the difference of the maximum and minimum values, \( (1/2)(71 - 47) = 12 \). The period would be (naturally) 24 hours, so the frequency \( (2\pi)/24 = \pi/12 \). The equation of a rough sinusoidal model might be \( 59 + 12 \sin \left( \frac{\pi}{12} (t - 9) \right) \). One could then tweak the model to get a closer correspondence to the observed data.
20. a. A sinusoidal model is of the form $\sin(B(t-C))$, where $t$ represents the month number. The period of any sinusoidal model for monthly temperature is certainly going to be 12 months, thus $B = 2\pi/12 = \pi/6 = 0.5236$. It is reasonable to take the average of all recorded high temperatures as the vertical shift in the sinusoidal model. This gives $D = 70.5$. We can estimate twice the amplitude as the difference between the maximum and minimum values, 77.6 and 64.4. This yields $A = 13.2/2 = 6.6$. The phase shift seems to place the “start” of a cycle at about May 22, which is around month number 5.7. This gives $C = 5.7$ and establishes $70.5 + 6.6 \sin(0.5236(t-5.7))$ as one of many reasonable models.

b. The atmosphere itself is a reservoir that delays the response to radiative heating.

21. We assume a model of the form $D + A \sin(B(t-C))$, where we take $t$ to represent the day of the year and assume a year of 365 days. The natural period is 365 days, which gives $B = 2\pi/365$. The average of all measured temperatures is 76˚. This gives $D = 76$. The maximum and minimum temperatures are 99˚ and 53˚ respectively. Each differs by 23˚ from the average. Based on this information we take $A = 23$. Determining the phase shift $C$ is more delicate. We can estimate that the daily high temperature was 76 (sin ($B(t-C) = 0$ somewhere between $t = 91$ and $t = 105$. Over this period of fourteen days, the daily high temperature rose five degrees from 72˚ to 77˚, so we can assume that the temperature rose one degree every three days. Using this we estimate that the Dallas daily high temperature hit 76˚ on or about day 88. That is, we take $C = 88$. The formula we have constructed is $76 + 23 \sin(\frac{2\pi}{365}(t-88))$. In order to arrive at this formula, we implicitly assumed that we recorded the high and low temperatures of the year on the days sampled. This is unlikely, and the formula can be tweaked experimentally from here.

22. A sinusoidal model is of the form $D + A \sin(B(t-C))$, where $t$ is the month number. The period of any annual phenomenon is 12 months. Thus we take $B = 2\pi/12 = 0.5326$. It is reasonable to take the average number of tornadoes as the vertical shift in the sinusoidal model. This gives $D = 71.9$. We can estimate twice the amplitude as the difference between the maximum and minimum, 191 and 16. This yields $A = 87.5$. The phase shift seems to place the “start” of a cycle at about March 15, that is around month number 3.5. This gives $C = 3.5$ and establishes $71.9 + 87.5 \sin(0.5236(t-3.5))$ as our first model. One obvious defect in this model is that it predicts a negative number of tornadoes for December.

23. The function $R(t) = 0.85 \sin(\frac{2\pi}{5}t)$ has a period of 5 seconds, so in 1 minute a person breathes $\frac{60}{5} = 12$ seconds.

24. a. The length of the planet’s year appears to be 26 or 27 days.

b. Substituting $t = 26$ into $D = 1.818t^{2/3}$, we find that the planet orbits roughly 16 million miles from its pulsar.

c. The orbital speed of the planet is $(2\pi \times 16,000,000)/26 = 3,866,000$ miles per day.

d. The orbital speed of Earth is only $(2\pi \times 93,000,000)/365 = 1,600,000$ miles per day.

25. a. With $t$ in seconds, the period is 30 seconds.
b. Height is given by \(110 + 100 \sin \left( \frac{\pi}{15} (t - 7.5) \right)\).

c. The horizontal distance is given by \(x = -100 \sin \left( \frac{\pi}{15} t \right)\).

d. The intervals of \(t\) for which you are moving forward are \(7.5 < t < 22.5, 37.5 < t < 22.5\), and so on. Assuming that you are facing to your right, you will be moving forward on the top half of the cycle.

e. \(x = 100 \sin \left( \frac{\pi}{15} t \right)\); for the intervals \(-7.5 < t < 7.5, 22.5 < t < 37.5\), and so on.

f. \((y-110)^2 + x^2 = 10,000\).

26. The calculations are similar to those in Problem 25: The amplitude is half the diameter of the wheel, or 6 meters, the vertical shift is 6 + 2 = 8 meters, and the period is 20 seconds. If we take \(t = 0\) when the person we’re charting is at the bottom and use a cosine function, a formula would be \(H(t) = 8 + 6 \cos \left( \frac{\pi}{10} (t - 10) \right)\).

27. The average brightness is \(\frac{4.35 + 3.65}{2} = 4\) and the amplitude of the variation in \(\frac{4.35 - 3.65}{2} = 0.35\). Using a cosine function and shifting by half a period to put the minimum at \(t = 0\) we get \(B(t) = 4 + 0.35 \cos \left( \frac{2\pi}{5.4} \left(t - \frac{5.4}{2} \right) \right)\) or \(B(t) = 4 + 0.35 \cos \left( \frac{\pi}{2.7} \left(t - 2.7 \right) \right)\). Using a sine function, we would get \(B(t) = 4 + 0.35 \sin \left( \frac{\pi}{2.7} (t - 1.35) \right)\).

28. a. After twenty years Tony will have lived through \(365 \times 20 + 5 = 7305\) days, since he will have celebrated 5 leap years. Any appropriate range of days must include days 7306 through 7365.

b. January 15 and February 7 are good days for Tony to compete.

c. January 10 and February 7 would be recommended days for love.

d. January 29 would certainly be the best day for Tony to take an exam.
e. The period around February 15 is likely to be a disaster for poor Tony.
f. February 3 should be Tony’s best day.

29. a. One possible model based on the scatter plot is that the average global temperature is oscillating above and below the linear trend with an amplitude of about 0.1° and a period of about 60 years. A corresponding sine function would be \( E(t) = 0.1\sin\left(\frac{2\pi}{60}(t-45)\right) \), where the phase shift value 45 is chosen so that the fluctuation \( E \) peaks at \( t = 60 \).
b. If we add this to the linear fit, the plot looks like this, over the same range as shown in the scatter plot:

c. For the year 2005 this function predicts an average global temperature of 0.0042(125) \( +14.67 + E(125) = 15.3° \).

Section 7.3  Solving Equations with Sine and Cosine: The Inverse Functions

1. The number of hours of daylight in San Diego is approximated by \( H = 12 + 2.4\sin\left(\frac{2\pi}{365}(t-80)\right) \).

   For \( H = 11 \) hours:
   \[
   11 = 12 + 2.4\sin\left(\frac{2\pi}{365}(t-80)\right)
   \]
   \[
   -0.4167 = \sin\left(\frac{2\pi}{365}(t-80)\right)
   \]
   \[
   \arcsin(-0.4167) = \arcsin\left(\sin\left(\frac{2\pi}{365}(t-80)\right)\right)
   \]
   \[
   \arcsin(-0.4167) = \frac{2\pi}{365}(t-80)
   \]
   \[
   t-80 = \frac{\arcsin(-0.4167) \cdot 365}{2\pi}
   \]
   \[
   t = 55
   \]

   Using the symmetry of the sine function, there is a second solution:
   \[
   \pi - \arcsin(-0.4167) = \frac{2\pi}{365}(t-80)
   \]
   \[
   t-80 = \frac{\left(\pi - \arcsin(-0.4167)\right) \cdot 365}{2\pi}
   \]
   \[
   t = 287
   \]

San Diego has 11 hours of daylight on Feb 24 and Oct 14.
For $H = 10$ hours:  

$10 = 12 + 2.4 \sin \left( \frac{2\pi}{365} (t - 80) \right)$

$-0.8333 = \sin \left( \frac{2\pi}{365} (t - 80) \right)$

$t - 8 = \frac{\arcsin (-0.8333) \cdot 365}{2\pi}$

$t = 23$

or

$t - 80 = \frac{\pi - \arcsin (-0.8333) \cdot 365}{2\pi}$

$t = 320$

San Diego has 10 hours of daylight on Jan 23 and Nov 16.

For $H = 9$ hours:  

$9 = 12 + 2.4 \sin \left( \frac{2\pi}{365} (t - 80) \right)$

$-1.25 = \sin \left( \frac{2\pi}{365} (t - 80) \right)$

$\arcsin (-1.25) = \frac{2\pi}{365} (t - 80)$

There is no solution because $\arcsin (-1.25)$ is undefined ($\sin x$ can never be $-1.25$).

2. a. The height of the water is given by $h(t) = 10 + 4 \sin \left( \frac{\pi t}{6} \right)$, where $t$ is the number of hours past midnight.

High tide occurs when $\sin \left( \frac{\pi t}{6} \right) = 1$; that is, when $\frac{\pi t}{6} = \frac{\pi}{2}$. Solving for $t$ gives $t = 3$ hours. Since high tide occurs every 12 hours in the course of a 24-hour day, high tide occurs in this case at 3 AM and 3 PM.

b. Low tide occurs when $\sin \left( \frac{\pi t}{6} \right) = -1$; that is, when $\frac{\pi t}{6} = \frac{3\pi}{2}$. Solving for $t$ gives $t = 9$ hours. Low tide occurs at 9 AM and 9 PM.

c.  

$h(t) = 10 + 4 \sin \left( \frac{\pi t}{6} \right)$

$-0.5 = 10 + 4 \sin \left( \frac{\pi t}{6} \right)$

$\arcsin \left( \frac{-1}{2} \right) = \frac{\pi t}{6}$

$t = 7$ hours

7:00 AM and 7:00 PM.

$h(t) = 10 + 4 \sin \left( \frac{\pi t}{6} \right)$

$0 = 10 + 4 \sin \left( \frac{\pi t}{6} \right)$

$\arcsin (0) = \frac{\pi t}{6}$

$t = 6$ hours

6:00 AM and 6:00 PM.
Solving Equations with Sine and Cosine

Section 7.3

1. \[ h(t) = 10 + 4 \sin \frac{\pi t}{6} \]

2. \[ h(t) = 12 + 4 \sin \frac{\pi t}{6} \]

3. \[ \frac{1}{4} = \sin \frac{\pi t}{6} \quad \text{arcsin} \left( \frac{1}{4} \right) = \frac{\pi t}{6} \]

4. \[ \frac{1}{2} = \sin \frac{\pi t}{6} \quad \text{arcsin} \left( \frac{1}{2} \right) = \frac{\pi t}{6} \]

5. \[ \arcsin \left( \frac{1}{4} \right) = \frac{\pi t}{6} \]

6. \[ \arcsin \left( \frac{1}{2} \right) = \frac{\pi t}{6} \]

7. \[ 0.25268 = \frac{\pi t}{6} \quad t = 1 \text{ hour} \]

8. \[ 1:00 \text{ AM and 1:00 PM.} \quad 12:30 \text{ AM and 12:30 PM.} \]

3. a. The temperature reaches a maximum at \( 69^\circ + 3^\circ = 72^\circ \).
   b. The temperature reaches a minimum at \( 69^\circ - 3^\circ = 66^\circ \).
   c. \[ T(t) = 69 + 3 \sin \frac{\pi t}{10} \]

4. a. The average temperature will be \( 67^\circ \) and the amplitude of the changes will be \( 3^\circ \). With \( t \) in minutes and \( t = 0 \) at noon, an equation is \[ T(t) = 67 + 3 \cos \left( \frac{2\pi t}{15} \right) \].
   b. \( 66^\circ \) is 1 degree below the average, so it occurs when \( -1 = 3 \cos \left( \frac{2\pi t}{15} \right) \) or \( \frac{2\pi t}{15} = \arccos \left( -\frac{1}{3} \right) = 1.991 \).

Thus the first time the temperature reaches \( 66^\circ \) is \( 1.991 \times \frac{15}{2\pi} = 4.75 \) minutes after noon. The temperature will pass \( 66^\circ \) on the way up during this same cycle, as far after the minimum at 7.5 minutes as this first time is before the minimum. So this second \( 66^\circ \) time is \( 7.5 + (7.5 - 4.75) = 10.25 \) minutes. To find the remaining \( 66^\circ \) times between noon and 1 PM we add 15 minutes, 30 minutes, and 45 minutes to the two times we have found so far, 4.75 minutes and 10.25 minutes.
5. The model we developed for the tides at the Bay of Fundy was 
   \( 25 + 25 \sin(\frac{2\pi}{11}t) \).
   
   a. In our model, low tide corresponds to 
   \( 25 + 25 \sin(\frac{2\pi}{11}t) = 0 \), hence to \( t = 8.25 \) hours. Solving 
   \( 25 + 25 \sin(\frac{2\pi}{11}t) = 5 \) yields \( t = 9.38 \) hours as the first solution after 8.25. It takes roughly one hour 
   for the water to rise 5 feet after low tide.
   
   b. Average tide occurs in our model at \( t = 0 \), at which point the depth of water is measured as 25 feet.
   
6. Our formula from Problem 26 in Section 7.2 is 
   \( H(t) = 8 + 6 \cos\left(\frac{\pi}{10}(t-10)\right) \). When
   
   \( 10 = 8 + 6 \cos\left(\frac{\pi}{10}(t-10)\right) \), we have \( \frac{1}{3} = \cos\left(\frac{\pi}{10}(t-10)\right) \) and \( \frac{\pi}{10}(t-10) = \arccos\left(\frac{1}{3}\right) = 1.231 \). One solution
   is \( t = 1.231 \cdot \frac{10}{\pi} + 10 = 13.92 \). Since the period is 20 seconds, this 10-meter time is past the peak at 10 
   seconds, so there is an earlier 10-meter time at \( 20 - 13.92 = 6.08 \) seconds. The remaining times as the 
   wheel continues to turn can be found by adding multiples of 20 seconds to 6.08 seconds and 13.92 seconds.

7. Our Fairbanks high temperature function from Problem 15 in Section 7.2 is 
   \( T(t) = 22 + 42 \sin\left(\frac{2\pi}{365}(t-131)\right) \), with \( t \) in days from the beginning of the year. \( T \) will be 0 when
   
   \( \sin\left(\frac{2\pi}{365}(t-131)\right) = -\frac{22}{42} = -0.524 \). Then \( \frac{2\pi}{365}(t-131) = \arcsin(-0.524) = -0.552 \) and
   
   \( t = \left(-0.552 \cdot \frac{365}{2\pi} + 131\right) = 98.9 \). Thus the maximum daily temperature should be 0° around the 99th day, or 
   April 9. There is another solution later in the year when temperatures are on the way back down. Recall 
   that the coldest day of the year is day 40. Our second 0° day will be as far before the coldest day next year 
   as day 99 is after the coldest day this year. Thus another solution is \( (365 + 40) - (99 - 40) = 346 \) or 
   December 12.

8. Our model from Problem 19 of Section 7.2 is \( T(t) = 59 + 12 \sin\left(\frac{\pi}{12}(t-9)\right) \). This model indicates that the
   temperature is at its average at 9 AM and 9 PM, with a maximum at 3 PM and a minimum at 3 AM.
   
   a. \( 50 = 59 + 12 \sin\left(\frac{\pi}{12}(t-9)\right) \) when \( \sin\left(\frac{\pi}{12}(t-9)\right) = \frac{50-59}{12} = -\frac{3}{4} \). Then
   
   \( \frac{\pi}{12}(t-9) = \arcsin\left(-\frac{3}{4}\right) = -0.848 \) and \( t = -0.848 \cdot \frac{12}{\pi} + 9 = 5.76 \), around 5:45 AM. The same temperature 
   is reached at night as temperatures are decreasing to the minimum at 3 AM. This happens at 
   \( 3 - (5.76 - 3) = 0.24 \) or about 15 minutes after midnight.
   
   b. For a temperature of 60° the calculations are as follows:
   
   \( \sin\left(\frac{\pi}{12}(t-9)\right) = \frac{60-59}{12} = \frac{1}{12} \)
   
   \( \frac{\pi}{12}(t-9) = \arcsin\left(\frac{1}{12}\right) = 0.0834 \)
   
   \( t = 0.0834 \cdot \frac{12}{\pi} + 9 = 9.32 \)
   
   This value of \( t \) corresponds to about 9:20 AM. The temperature will also be 1° above its average on the 
   way back down, at 20 minutes before 9 PM, that is, at 8:40 PM.
9. The function we constructed in Problem 20 of Section 7.2 is \( T(t) = 70.5 + 6.6 \sin(0.5236(t - 5.7)) \), with \( t \) in months, \( t = 0 \) corresponding to the beginning of January.
   
   a. \( 65 = 70.5 + 6.6 \sin(0.5236(t - 5.7)) \)
   \[
   \sin(0.5236(t - 5.7)) = \frac{65 - 70.5}{6.6} = -0.8333 \\
   0.5236(t - 5.7) = \arcsin(-0.8333) = -0.9851 \\
   t = \frac{-0.9851 + 5.7}{0.5236} = 3.82
   
   This solution is near the end of April. Another solution is located symmetrically to the left of the minimum at 2.7, namely at \( 2.7 - (3.82 - 2.7) = 1.58 \), in the middle of February.

   b. \( 70 = 70.5 + 6.6 \sin(0.5236(t - 5.7)) \)
   \[
   \sin(0.5236(t - 5.7)) = \frac{70 - 70.5}{6.6} = -0.0758 \\
   0.5236(t - 5.7) = \arcsin(-0.0758) = -0.0759 \\
   t = \frac{-0.0759 + 5.7}{0.5236} = 5.56
   
   This solution is in mid-May. There is another solution located symmetrically on the other side of the maximum at 8.7, namely at \( 8.7 + (8.7 - 5.56) = 11.84 \), near the end of December.

   c. According to the model, the average daytime high does not reach 80°.

10. Assuming that the wall is at least 25 feet tall, that the wall rises vertically from a perfectly flat and horizontal ground, that there are 25 unobstructed feet of this perfect surface extending out from the wall, and that the ladder is so rigid it never sags—these are all naturally satisfied in Trigland, the place where this sort of problem is solved—the equation relating \( x \) to \( \theta \) is \( \cos \theta = \frac{x}{25} \). Expressing \( \theta \) as a function of \( x \), we get \( \theta = f(x) = \arccos(x/25) \) for \( 0 \leq x \leq 25 \). Examining the accompanying plot, we find that \( f \) is concave down everywhere and has its maximum at \( x = 0 \).

   ![Graph](image_url)

**Exercising Your Algebra Skills**

1. No solutions: \( |\sin x| \leq 1 \) for all \( x \).

2. Basic solution: \( \theta = \arcsin(0.4) = 0.412 \)
   
   General solution: \( \theta = 0.412 + 2n\pi \) and \( \theta = (\pi - 0.412) + 2n\pi = 2.730 + 2n\pi \)

3. No solutions: \( |\sin x| \leq 1 \) for all \( x \).

4. Basic solution: \( \theta = \arcsin(0.75) = 0.848 \)
   
   General solution: \( \theta = 0.848 + 2n\pi \) and \( \theta = (\pi - 0.848) + 2n\pi = 2.294 + 2n\pi \)

5. Basic solution: \( \theta = \arcsin(-0.75) = -0.848 \)
   
   General solution: \( \theta = -0.848 + 2n\pi \) and \( \theta = (\pi - (-0.848)) + 2n\pi = 3.990 + 2n\pi \)
6. Basic solution: \(2x = \arcsin(0.6) = 0.644\)
   General solution: \(2x = 0.644 = 2n\pi\) and \(2x = (\pi - 0.644) + 2n\pi = 2.498 + 2n\pi\)
   Thus, \(x = 0.322 + n\pi\) and \(x = 1.249 + n\pi\)

7. Basic solution: \(2x = \arccos(-0.6) = 2.214\)
   General solution: \(2x = 2.214 + 2n\pi\) and \(2x = (2\pi - 2.214) + 2n\pi = 4.069 + 2n\pi\)
   Thus, \(x = 1.107 + n\pi\) and \(x = 2.034 + n\pi\)

8. Basic solution: \(x = \arccos\left(\frac{2}{3}\right) = 0.841\)
   General solution: \(x = 0.841 + 2n\pi\) and \(x = (2\pi - 0.841) + 2n\pi = 5.442 + 2n\pi\)

9. Using a double-angle formula from Section 8.1, we can write this equation as
   \(5(2\sin x \cos x) = 3\cos x\) or \(10\sin x \cos x = 3\cos x\). Now either \(\cos x = 0\), which happens when
   \(x = \frac{\pi}{2} + n\pi\), or we can divide through by \(\cos x\) to get \(\sin(x) = 0.3\). This equation has basic solution
   \(x = \arcsin(0.3) = 0.305\) and general solution \(x = 0.305 + 2n\pi\) and \(x = (\pi - 0.305) + 2n\pi = 2.837 + 2n\pi\).
   Thus the complete solution is \(x = \frac{\pi}{2} + n\pi, 0.305 + 2n\pi, 2.837 + 2n\pi\).

Section 7.4 The Tangent Function

1. a. \(y = \tan(2x)\)

   ![Graph of y = tan(2x)]

b. \(y = \tan\left(\frac{1}{2}x\right)\)

   ![Graph of y = tan(1/2x)]
The Tangent Function  
Section 7.4 

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c. \( y = 3 \tan x \)

d. \( y = -2 \tan x \)

e. \( y = \tan(x - 30) \)

f. \( y = -10 + \tan(x) \)

2. We write these functions for angles in radians.
   a. \( y = \tan(\pi x) \)
   b. \( y = 25 + \tan x \)
   c. \( y = \tan(2x) \)

3. The standard tangent graph has asymptotes at \( \theta = \pm \pi / 2 \) and the table indicates asymptotes at \( \pm \pi / 3 \), so we need to compress the graph horizontally by a factor of 3/2. The equation is \( y = \tan\left(\frac{3}{2}x\right) \). This checks with the other values in the table.
Looking at the triangle formed by the ground and the base of the Statue of Liberty we find \( \tan \alpha = \frac{46}{x} \). Considering the triangle whose vertices are the observer, the point of the base closest to the observer and the top of the Statue of Liberty we see that \( \tan (\alpha + \theta) = \frac{92}{x} \). These equations may be recast as \( \alpha = \arctan \left( \frac{46}{x} \right) \) and \( \alpha + \theta = \arctan \left( \frac{92}{x} \right) \), which gives us the function \( \theta(x) = \arctan \left( \frac{92}{x} \right) - \arctan \left( \frac{46}{x} \right) \). Examining the graph of \( \theta(x) \), we find a maximum value of 19° 30' at a distance of 65 meters.

The height \( b \) of the building is given by \( \tan 32^\circ = \frac{b}{60} \). Thus \( b = 60 \tan 32^\circ = 37.5 \) meters. The distance \( d \) from the top of the smokestack to the ground is \( d = 60 \tan 54^\circ = 82.6 \) meters. The height of the smokestack is \( d - g = 45.1 \) meters.

6, 7. The slope \( m \) of a line is computed as rise/run. However, for a line through the origin, the line will pass through the point \((\cos \theta, \sin \theta)\), where \( \theta \) is the angle formed by the given line and the \( x \)-axis. Using this point and the origin to compute the slope we get \( m = \frac{\sin \theta}{\cos \theta} = \tan \theta \). The angles formed by the lines \( y = x, y = 2x, y = 3x, y = 4x \), are \( \pi/4, \arctan(2), \arctan(3), \) and \( \arctan(4) \), respectively. The line whose equation is \( y = mx + b \) may be viewed as the translation by \( b \) units in the \( y \)-direction of the line \( y = mx \). Since the angle formed by a line and the \( x \)-axis is not changed by a translation of the line, we may always view the slope of a line as the tangent of the angle formed by the line and the \( x \)-axis.
8. \[ \frac{1}{\tan x} \]

9. \[ \frac{1}{\tan x} \]

10. Looking at the graph of \( \tan x \) we can make some observations about \( \frac{1}{\tan x} \).

Since \( \tan x \) has zeros at \( x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \), \( 1/\tan x \) will have vertical asymptotes at these values of \( x \).

Since \( \tan x \) has vertical asymptotes at \( x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots \), \( 1/\tan x \) will have zeros at these values.

Further, since \( \tan x \) is increasing on each of its branches, we expect to find \( 1/\tan x \) decreasing on each of its own branches.
11. a. The angle from the horizontal to the top of the painting is \( \arctan \left( \frac{5}{x} \right) \) and the angle to the bottom is \( \arctan \left( \frac{1}{x} \right) \), so \( \beta = \arctan \left( \frac{5}{x} \right) - \arctan \left( \frac{1}{x} \right) \).

b. The graph indicates a maximum angle at a distance from the wall of about 2.4 or 2.5 feet. The maximum angle is close to 45°.

12. a. \( \alpha = \arctan \left( \frac{y}{500} \right) \)

b. \( \alpha = \arctan \left( \frac{1000}{500} \right) = 63.4° \)

c. \( \alpha = \arctan \left( \frac{2000}{500} \right) = 76.0° \)

13. a. \( \nu \)  0  0.5\( c \)  0.9\( c \)  0.95\( c \)  0.99\( c \)  0.999\( c \)

\( M \)  1  1.155  2.294  3.203  7.089  22.366
The actual function giving mass can be reasonably approximated by \( M = 1 + \frac{\tan\left(\frac{\pi v}{2}\right)}{6} \).

**Exercising Your Algebra Skills**

1. Basic solution: \( \theta = \arctan\left(\frac{5}{4}\right) = 0.896 \)
   General solution: \( \theta = 0.896 + n\pi \)

2. Basic solution: \( \theta = \arctan\left(\frac{4}{5}\right) = 0.675 \)
   General solution: \( \theta = 0.675 + n\pi \)

3. Basic solution: \( \theta = \arctan(-1) = -\frac{\pi}{4} \)
   General solution: \( \theta = -\frac{\pi}{4} + n\pi \), which we can also write as \( \theta = \frac{3\pi}{4} + n\pi \).

4. Basic solution: \( \theta = \arctan(2) = 1.107 \)
   General solution: \( \theta = 1.107 + n\pi \)

5. Basic solution: \( \theta = \arctan\left(\frac{4}{5}\right) = 0.675 \)
   General solution: \( \theta = 0.675 + n\pi \)

6. Basic solution: \( \theta = \arctan\left(\frac{5}{4}\right) = 0.896 \)
   General solution: \( \theta = 0.896 + n\pi \)

7. \( \sin \theta + \cos \theta = 0 \Rightarrow \sin \theta = -\cos \theta \Rightarrow \theta = \arctan(-1) = -\frac{\pi}{4} \)
   General solution: \( \theta = -\frac{\pi}{4} + n\pi \), which can be written as \( \theta = \frac{3\pi}{4} + n\pi \).

8. \( 4\sin \theta = 3\cos \theta \Rightarrow \theta = \arctan\left(\frac{3}{4}\right) = 0.644 \)
   General solution: \( \theta = 0.644 + n\pi \)
Chapter 7 Review Problems

1. The answers are the same because \( \cos x = \sin \left(x + \frac{\pi}{2}\right) \) and \( 5.70 + \frac{\pi}{2} = 7.2708 \).

2. a. \( \theta = 120^\circ, 420^\circ, 480^\circ, \ldots \), \( \theta = -240^\circ, -300^\circ, -600^\circ, -660^\circ, \ldots \)
   b. \( t = 2\pi/3, 7\pi/3, 8\pi/3, \ldots \), \( t = -4\pi/3, -5\pi/3, -10\pi/3, \ldots \)

3. a. \( \theta = 315^\circ, 405^\circ, 675^\circ, \ldots \), \( \theta = -45^\circ, -315^\circ, -405^\circ, -675^\circ, \ldots \)
   b. \( \theta = 7\pi/4, 9\pi/4, 15\pi/4, \ldots \), \( \theta = -\pi/4, -7\pi/4, -9\pi/4, -15\pi/4, \ldots \)

<table>
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<th>vertical shift</th>
<th>amplitude</th>
<th>frequency</th>
<th>period</th>
<th>phase shift</th>
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<td>40</td>
<td>2\pi/83</td>
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</table>

12–19. Answers will vary.

20. The side view of the swing is shown. For any \( \theta \) between 0° and 60°, or 0 and \( \frac{\pi}{3} \) radians, the hypotenuse is 8 ft because that is the length of the chain. At the maximum, when \( \theta = \frac{\pi}{3} \), the vertical side of the triangle shown is \( 8 \cos \frac{\pi}{3} = 4 \) ft and the base of the triangle is 693 ft.

![Diagram of the swing](image)

a. For the vertical motion, the height of the seat oscillates between 3 feet above ground level and 11 – 4 = 7 feet, so it is centered at a height of 5 feet above ground level. Since this happens over a 3-second interval, one sinusoidal model for the height is \( y = 5 + 2 \cos \left(\frac{2\pi}{3} t\right) \), where \( t \) is in radians, which starts at the maximum height of 7 feet when \( t = 0 \).
b. For the horizontal motion, the distance left and right of center oscillates between −6.93 and 6.93 feet, with an average value of 0. However, during 3 seconds this motion completes only half a cycle (say from the far right to the far left). Thus, one model is \( x = 6.93\cos\left(\frac{2\pi}{6}t\right) \), which starts at the maximum horizontal distance that corresponds to the maximum height of \( y = 7 \) at time \( t = 0 \). The 6 in the denominator gives the required period of 6 seconds.

21. If the bungee cord didn’t contract, the jumper would simply oscillate at a maximum of 40 feet above and below the 160-feet level. So a simple sinusoidal model would be \( D = 160 + 40\cos\frac{2\pi}{6}t \), which starts at a distance \( d = 200 \) feet below the bridge at time \( t = 0 \). To account for the decaying oscillation, which dies out over the first 10 cycles, we use a function of the form \( D = 160 + 40a\cos\frac{2\pi}{6}t \) for some \( a < 1 \). Suppose we interpret the information that the oscillation dies out after 10 cycles of length 6 seconds to mean that the distance after the tenth oscillation is within 0.5 foot of the 160-feet level mark since theoretically the jumper never comes to a complete stop. So, when \( t = 60 \), we need to solve the following equation:

\[
\begin{align*}
160 + 40a_{60}^6 \cos \frac{2\pi}{6} \cdot 60 & = 160.5 \\
160 + 40a_{60}^6 \cos 20\pi & = 160.5 \\
160 + 40a_{60}^6 & = 160.5 \\
40a_{60}^6 & = 0.5 \\
a_{60}^6 & = 0.0125 \\
a & = (0.0125)^{1/60} = 0.92957
\end{align*}
\]

Thus, our model is \( D = 160 + 40(0.92957)^t \cos \frac{\pi}{3}t \).

22. a. \( H = 1.5\sin\left(\frac{2\pi}{500}t\right) \)

b. \( H = 1.5 + 1.5\sin\left(\frac{2\pi}{500}t\right) \)

23. The pedals start at a height of \( 4 + 10/2 = 9 \) inches and the amplitude is 5 inches. The function might be \( H = 9 + 5\sin\left(\frac{2\pi}{3}t\right) \) if the child is going up at time 0.

24. a. If we add a uniform horizontal speed of 10 inches per second and take the midline as the average position of the child’s feet, the graph of \( y = 5\sin\left(\frac{2\pi}{3} \cdot \frac{x}{10}\right) \) might indicate the height \( y \) as a function of the distance \( x \) traveled, so that a plot of \( y \) versus \( x \) gives a picture of the path of the child’s feet through space.
b. The graph of the function over 10 seconds looks like this:

The graph makes it clear that our model is suspect: there is no indication in the graph that conditions are different when the stick is in contact with the ground.

c.

This graph looks a bit more realistic, but it is still wrong. Unless the stick slips on the floor, the child’s feet can’t move forward much while the stick is in contact with the floor (they can move forward a bit as the stick rotates around the contact point) but this graph still has uniform forward motion in the \( x \)-direction.

25. a. \( s = 4000 \cdot \frac{\pi}{4} = 3142 \) miles  
   b. \( s = 4000 \cdot \frac{\pi}{3} = 4189 \) miles  
   c. \( s = 4000 \cdot \frac{5\pi}{6} = 10,472 \) miles  
   d. \( s = 4000 \cdot 15^\circ \cdot \frac{\pi}{180^\circ} = 1047 \) miles

26. a. \( \frac{180 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ radians}}{\text{rev}} = 360\pi \frac{\text{radians}}{\text{min}} \)  
   b. \( \frac{2 \text{ ft}}{\text{radian}} \cdot \frac{360\pi \text{ radians}}{\text{min}} \cdot 1 \text{ min} = 720\pi \text{ feet} \)

27. Numerical answers will vary depending on the diameter \( D \). With \( D \) measured in inches, the tire makes \( \frac{5280 \cdot 12}{\pi D} \) revolutions per minute.

28.
29. a. Frequency $\frac{3}{4}$, period $\frac{8\pi}{3}$, amplitude 2, and phase shift 0.

b. Frequency $\frac{3}{4}$, period $\frac{8\pi}{3}$, amplitude 2, and phase shift $-\frac{4\pi}{3}$.

c. Frequency $\pi$, period $2$, amplitude 2, and phase shift $-\frac{3}{4\pi}$.

d. Frequency $\frac{3\pi}{4}$, period $\frac{8}{3}$, amplitude 2, and phase shift $\frac{1}{\pi}$.

30. a. $x = \pm \frac{4\pi}{9} + \frac{8\pi n}{3}$

b. $x = \pm \frac{4\pi}{9} + \frac{8\pi n}{3}$

c. $x = \frac{2}{3} - \frac{3}{4\pi} + 2n$ or $x = \frac{4}{3} - \frac{3}{4\pi} + 2n$

d. $x = \frac{8}{9} + \frac{1}{\pi} + \frac{8n}{3}$ or $x = \frac{16}{9} + \frac{1}{\pi} + \frac{8n}{3}$

31. a. 

The graph shows that arcsin(sin $x$) is not the same as the identity function $x$: Over the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the graph does look like $y = x$, but since the values of the inverse sine function are always between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inputs to the sine function outside this range will get moved back into this range by the function arcsin(sin $x$).
32.  
   a.  \( \theta = \arctan \frac{3}{0.98} + n\pi = 0.98 + n\pi \)  
   b.  \( \theta = \arctan 2 + n\pi = 1.11 + n\pi \)  
   c.  \( \theta = \arctan 7 + n\pi = 1.43 + n\pi \) 

33.  
   a.  \( x = 4.45522 \)  
   b.  \( x = 0.47943 \)