Calculus III: Sequences and Series Notes (Rigorous Version)

Logic

Definition ***(Proposition)*** A *proposition* is a statement which is either true or false.

Questions and commands are never propositions, but statements like "My Buick is maroon" (T) and "My Buick is black" (F) are propositions. More importantly for the purposes of this course, statements like "(a_n) converges to ℓ" are propositions, whose truth value you are asked to either find or verify.

Definition ***(Conditional Proposition)*** A *conditional proposition* is a proposition of the form "If p, then q" where p and q are propositions called the *hypothesis* and *conclusion*, respectively. We write $p \Rightarrow q$. A conditional proposition is true if it is always true (that is, for every possible instance of a true hypothesis, the conclusion holds true), and false if there is a counterexample.

An example of a conditional proposition is "If I am at WOU, then I am in Oregon." The hypothesis p is in this case the proposition "I am at WOU" and the conclusion q is "I am in Oregon." Note that for this particular proposition, although $p \Rightarrow q$ is true, $q \Rightarrow p$ is false.

The theorems you use in this course are conditional propositions that are true.

Definition ***(Converse)*** Let $p \Rightarrow q$ be a conditional proposition. The conditional proposition $q \Rightarrow p$ is called the *converse* of $p \Rightarrow q$.

As we saw above, a conditional proposition and its converse may have different truth values.

Definition ***(Contrapositive)*** Let $p \Rightarrow q$ be a conditional proposition. The *contrapositive form*, or simply the *contrapositive*, of $p \Rightarrow q$ is the conditional proposition "If not q, then not p" and is denoted $\neg q \Rightarrow \neg p$.

The contrapositive of "If I am at WOU, then I am in Oregon" is "If I am not in Oregon, then I am not at WOU." If you think about it, you should be able to convince yourself that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ are just two different ways of saying *the same thing*, and thus must have the same truth values. This is correct, and will have significant value to us in this course.

Theorems and Definitions for Sequences

Definition ***(Sequence; Subsequence)*** A *sequence* is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$, and can be thought of as an infinite ordered list of numbers. We denote a sequence by (a_n) and the n^{th} term of the sequence (a_n) by a_n. Given a sequence (a_n), a *subsequence* is the same function but whose domain is now an infinite subset of $\mathbb{N}$. A subsequence can be thought of as an infinite sub-list of the infinite list that comprises the original sequence.

Note that every sequence is trivially a subsequence of itself. A less trivial example is that $\left(\frac{1}{n^2}\right)$ can be interpreted as a subsequence of $\left(\frac{1}{n}\right)$, where the domain is no longer $\mathbb{N}$, but all natural numbers which are perfect squares. Also, $((-1)^n)$ is a subsequence of $\left(\cos\left(\frac{n\pi}{3}\right)\right)$, where the domain has been restricted from $\mathbb{N}$ to natural numbers of the form $3n$, $n \in \mathbb{N}$.

Definition ***(Tail)*** Given a sequence (a_n) and a fixed $k \in \mathbb{N}$, a *tail* is a particular kind of subsequence of (a_n) of the form $(a_{n+k}) = (a_{n+k})_{n\in\mathbb{N}}$.

Definition ***(Convergence)*** Let (a_n) be a sequence. Then we say that (a_n) *converges to* ℓ if, no matter how small we choose a positive real number ε, we can find a $N \in \mathbb{N}$ so that all of the terms in the tail (a_{n+N}) are in the interval $(\ell - \varepsilon, \ell + \varepsilon)$. *Note that N will typically be larger for smaller ε.* If (a_n) converges to ℓ, we may write either $(a_n) \to \ell$ or $\lim_{n\to\infty} (a_n) = \ell$.

Theorem Const: Let (a_n) be a sequence. If $a_n = c \in \mathbb{R}$ for every $n \in \mathbb{N}$, then $(a_n) \to c$.

Theorem 8.1.2: If f is a function with $\lim_{x\to\infty} f(x) = \ell \in \mathbb{R}$, and if (a_n) is a sequence given by $a_n = f(n)$ for each $n \in \mathbb{N}$, then $\lim_{n\to\infty} (a_n) = \ell$.

Note that the converse of this theorem is false, because one can't control how the function behaves between the integers.

Theorem 8.1.2.5: If $p > 0$, then $\left(\dfrac{1}{n^p}\right) \to 0$.

Natural Log Theorem (NLT): If $p > 0$, then $\left(\dfrac{1}{(\ln n)^p}\right)_{n=2}^{\infty} \to 0$.

Limit Laws: Suppose (a_n) and (b_n) are convergent sequences converging to ℓ and m respectively, and

$c \in \mathbb{R}$. Then

$$(a_n \pm b_n) \to \ell \pm m \tag{1}$$
$$(a_n b_n) \to \ell m \tag{2}$$
$$(c a_n) \to c\ell \tag{3}$$
$$\left(\frac{a_n}{b_n}\right) \to \frac{\ell}{m} \qquad \text{provided that } b_n \neq 0 \text{ for all } n \in \mathbb{N}, \text{ and } m \neq 0. \tag{4}$$

The next theorem includes Theorem 8.1.4, but indicates that the converse is true as well. The converse is useful for applying the Test for Divergence, to appear in §8.2.

Theorem 8.1.4. (Slightly Improved): Let (a_n) be a sequence. If $(|a_n|) \to 0$ then $(a_n) \to 0$. Also, if $(a_n) \to 0$ then $(|a_n|) \to 0$.

Definition ***(Boundedness)*** Let (a_n) be a sequence. If there exists $M > 0$ such that all of the terms of (a_n) are in $[-M, M]$, then we say that (a_n) is bounded.

Definition ***(Unboundedness)*** Let (a_n) be a sequence. If no closed interval contains the range of (a_n), then we say that (a_n) is unbounded.

Recall that we said that if a sequence doesn't converge, the sequence is said to *diverge*. There are two basic types of divergent sequences, *bounded divergent* and *unbounded divergent*.

An example of a bounded divergent sequence is $((-1)^n)$, while an example of an unbounded divergent sequence is (n^2). Our goal is to develop two tools to show that divergent sequences are in fact divergent. The first will help us show that certain bounded divergent sequences diverge, while the second will help us show that certain unbounded divergent sequences diverge.

Geometric sequences are a very important type of sequence; the next theorem regards which geometric sequences converge.

Theorem 8.1.6: Let $(cr^n)_{n=0}^{\infty}$ be a geometric sequence with $c \neq 0$. Then

$$(cr^n) \begin{cases} \text{converges to } 0 & \text{if } |r| < 1 \\ \text{converges to } c & \text{if } r = 1 \\ \text{alternates between } -c \text{ and } c \text{ and thus diverges boundedly} & \text{if } r = -1 \\ \text{diverges unboundedly} & \text{if } |r| > 1 \end{cases}$$

The partitioning of divergence into bounded and unbounded categories is useful since it permits us to treat each rigorously. For each type of divergent sequence, a supplementary tool is provided for demonstrating divergence.

Subsequence Theorem: If $(a_n) \to \ell \in \mathbb{R}$, then every subsequence of (a_n) converges to ℓ.

Note that the contrapositive of this is "If not every subsequence of converges to ℓ, then $(a_n) \not\to \ell$." The latter form of the theorem is the most useful one in this course, though the non-contrapositive form is used in Monotonic Sequence Theorem problems at the end of §8.1.

Example: To show that $((-1)^n)$ does not converge, we need only note that the subsequence of $((-1)^n)$ where n is even is the constant sequence (1) and by Theorem Const converges to 1, while the subsequence of $((-1)^n)$ where n is odd is the constant sequence (-1) and by Theorem Const converges to -1. Thus by the Subsequence Theorem, $((-1)^n)$ diverges.

We'll assume that the following sequences are known unbounded divergent: (n^p) and $(\ln(n \pm k)^p)$ for $p > 0$, $k \in \mathbb{Z}$ and (a^n) for $a > 1$, and any positive constant multiple of any of these sequences.

Push Theorem: Let (a_n) be a known unbounded divergent sequence, and (b_n) another sequence. If eventually we have, for some positive constant c, $|b_n| \geq c\,a_n$, then (b_n) diverges.

Note that this is the divergent analog to the Squeeze Theorem.

Example: Suppose we are asked to show that $((-1)^n(n^3+2))$ diverges. We need only note that $|(-1)^n(n^3+2)| = n^3+2 \geq n^3$, which is a known divergent sequence, so by the Push Theorem, $((-1)^n(n^3+2))$ diverges.

Continuous Function Theorem (CFT) Let (a_n) be a sequence converging to $\ell \in \mathbb{R}$, and let $f(x)$ be a function that is defined on the range of (a_n) and which is continuous at ℓ. Then the sequence $(f(a_n))$ converges to $f(\ell)$.

Note that another way to state the CFT is that we can pass the limit through the function when the above hypotheses are satisfied; *i.e.*, if $(a_n) \to \ell$, if $f(a_n)$ is defined for each $n \in \mathbb{N}$, and if f is continuous at ℓ, then $\lim\limits_{n\to\infty} (f(a_n)) = f\left(\lim\limits_{n\to\infty} (a_n)\right)$.

A note of interest to those of you who are math majors: The CFT is actually a very good way to *define* the property of continuity for a function, and is in fact done so in Advanced Calculus.

Just so it is clear to which functions the CFT applies, we'll assume the following functions are continuous everywhere (*i.e.*, on all of $\mathbb{R}$): $\sin(x)$, $\cos(x)$, e^x, $x^p\,(p \geq 0)$, and $\tan^{-1}(x)$. Also, any sum or product of continuous functions is continuous (the quotient case is more complicated). Finally, $\tan(x)$ is continuous at any point not equal to $\frac{\pi}{2} + m\pi$, $m \in \mathbb{N}$, $\ln(x)$ is continuous on $(0, \infty)$, and x^p $(p < 0)$ is continuous on $\mathbb{R} \setminus \{0\}$.

Example: For example, to show that $\sin\left(\frac{(2n+1)\pi}{4n}\right) \to 1$, we first consider the argument

$$\frac{(2n+1)\pi}{4n} = \frac{(2+\frac{1}{n})\pi}{4} = \frac{2\pi + \pi\frac{1}{n}}{4}.$$

We know that $\left(\frac{1}{n}\right)$ converges to zero, since it is a p-sequence with $p = 1 > 0$. Thus by LL2, $\left(\pi\frac{1}{n}\right) \to \pi \cdot 0 = 0$. By Theorem Const, $(2\pi) \to 2\pi$ and $(4) \to 4$. Thus, by LL1 and LL4,

$$\left(\frac{(2n+1)\pi}{4n}\right) = \left(\frac{2\pi + \pi\frac{1}{n}}{4}\right) \to \frac{2\pi + 0}{4} = \frac{\pi}{2}.$$

Finally, by the CFT, $\sin\left(\frac{(2n+1)\pi}{4n}\right) \to \sin\left(\frac{\pi}{2}\right) = 1$.

Theorem ***(Decreasing Sequences by Derivative - DSBD)*** Let (a_n) be a sequence, and f a function such that $f(n) = a_n$ for all $n \in \mathbb{N}$. If for some $x_0 \geq 1$, we have $f'(x) < 0$ for all $x > x_0$, then (a_n) is decreasing for all $n > x_0$ (this means that (a_n) is eventually decreasing).[1]

Theorem ***(Log-Monomial-Exponential - LME)*** For all p, q, r, c_1, c_2, $c_3 > 0$ and for all x larger than some x_0 depending on the constants p, q, r, c_1, c_2, c_3 we have $c_1(\ln x)^p < c_2 x^q < c_3 e^{rx}$.

Theorem ***(Equal Sequences, Equal Limits - ESEL)*** If (a_n) and (b_n) are sequences such that $a_n = b_n$ eventually and $(a_n) \to \ell$, then $(b_n) \to \ell$.

Using l'Hôpitals's Rule for Sequences

The following algorithm is useful for the solutions of these problems, so we'll follow it for each such problem:

1. Identify the sequence formula as an indeterminate (if it is not an indeterminate form, you won't use l'Hôpital's Rule).

[1] Theorem DSBD is useful for showing that for example $\left(\frac{\ln n}{n}\right)$ is decreasing, in order to use the Alternating Series Test (which will appear a bit later on in the course) on $\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$.

2. If necessary, algebraically rearrange the sequence into the indeterminate form $\frac{\infty}{\infty}$ or $\frac{0}{0}$. For some problems (typically where n is present in both the base and the exponent), this may require taking the natural logarithm, so be prepared to invoke the CFT at the end of the problem.

3. In anticipation of using l'Hôpital's Rule, create the function formula for $f(x)$ from the sequence formula for a_n by substituting x for n.

4. Differentiate the numerator and denominator separately (no quotient rule!) and find the limit as $x \to \infty$ of the resulting function, if possible.

   (a) Repeat Step 4 as many times as necessary until a finite limit is found, if possible.

5. Write the conclusion of l'Hôpital's Rule *in the correct order.* It will appear something like this:

$$0 \quad = \quad \lim_{x\to\infty} \frac{1/x}{1} \quad \stackrel{l'H}{=} \quad \lim_{x\to\infty} \frac{\ln x}{x}.$$

6. Invoke the Asymptote Theorem (Theorem 8.1.2).

7. If necessary, algebraically undo Step 2 and invoke the CFT to find the original limit.

Using The Monotone Sequence Theorem to Demonstrate Convergence for Certain Recursively Defined Sequences

It's quite nice when one has an explicit formula for the n^{th} term of a sequence (a_n), but occasionally a sequence is defined by fixing the first term, then defining each subsequent term only in terms of the previous term (or some finite number of previous terms). An example is $a_1 = 4$; $a_{n+1} = 1 + \sqrt{a_n}$ for each $n \geq 1$. Such sequences do not easily submit to the tools we've developed thus far but there is a technique which will often work in these cases, involving the following important theorem.

Theorem (Monotone Sequence Theorem): Let (a_n) be a sequence. If (a_n) is eventually increasing or decreasing, and is bounded, then (a_n) converges.

The following are useful for MST problems:

Axiom 1: If $x, y \in \mathbb{R}^+$ (i.e., $x, y > 0$), $k > 0$ and $x < y$, then $x^k < y^k$.

Theorem Closed Interval - Sequence Range Limit Theorem (CISRL): Let (a_n) be a sequence, and $x, y \in \mathbb{R}$. If $x \leq a_n \leq y$ for all $n \in \mathbb{N}$ and $(a_n) \to \ell$, then $x \leq \ell \leq y$.

Definition Propositional Function: A propositional function is a collection of propositions defined on some domain, whose truth value may depend on values in that domain. For example $P(n) :=$"n is prime" is reasonably defined on $\mathbb{N}$ but true only on a strict subset of $\mathbb{N}$.

There are two hypotheses which must be satisfied to invoke the conclusion of the MST; boundedness and monotonicity of the sequence in question. Each of the hypotheses are generally verified by using the Principle of Mathematical Induction, which is based on demonstrating the truth of propositions the process of which follows:

1. Formulate the appropriate propositional function (like, for example $P(n) =$"$a_n < a_{n+1}$" or $P(n) =$"$a_n < 2$").
2. (Base Case - BC) Demonstrate the truth of $P(1)$.
3. (Inductive Hypothesis - IH) Write the hypothesis that there exists some $k \in \mathbb{N}$ such that $P(k)$ is true.
4. (Inductive Step - IS) Show, using the assumed truth of the IH, algebra, and arithmetic, that the conditional proposition "If $P(k)$ is true, then $P(k+1)$ is true" holds true.
5. (Conclusion) Invoke the Principle of mathematical Induction to conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

To show that an inductively defined sequence converges, we apply these steps twice (starting by proving monotonicity).

1. Identify which kind of monotonicity applies to the sequence (increasing or decreasing).
2. Use the five steps above to prove that your conjecture about monotonicity holds true.
3. If (a_n) is increasing then (a_n) is bounded below by a_1. In this case, choose a reasonable candidate for an upper bound for (a_n) and use the five steps above to show that (a_n) is bounded above by your candidate.
4. If (a_n) is decreasing then (a_n) is bounded above by a_1. In this case, choose a reasonable candidate for a lower bound for (a_n) and use the five steps above to show that (a_n) is bounded below by your candidate.
5. Having shown that (a_n) is both monotonic and bounded, invoke the conclusion of the MST; namely that (a_n) is convergent.

6. Find one or more candidates for the limit ℓ of (a_n) using Limit Laws, the Subsequence Theorem, and / or the CFT.

7. If more than one possible limit results from Step 6, rule the impossible candidate out by using boundedness and Theorem 8.1.8.

Series Definitions and Theorems

Definition ***(Series):*** Let (a_n) be a sequence. The *formal sum* $\sum_{n=1}^{\infty} a_n$ is called a series. We use the term "formal" at this point because we have not yet formulated what is meant by such an infinite sum.

The choice of starting index $n = 1$ is one of convenience; a series may just as well start at $n = 0$ or $n = 2$ or any other starting index, for that matter. In fact, there is nothing essential in any way about the starting index, since we can always start the series at whichever integer we choose by a change of variable. For this reason, we'll start most series at $n = 1$, with some notable exceptions.

Just as the most important thing about a sequence is whether or not the sequence converges, the same is true of any given series. Read on.

Definition ***(Convergence/Divergence of Series; 8.2.2 Improved):*** Let $\sum_{n=1}^{\infty} a_n$ be a series. Define the *associated sequence of partial sums* (s_n) by

$$s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n. \tag{1}$$

If $(s_n) \to s$ for some $s \in \mathbb{R}$, then we write that $\sum_{n=1}^{\infty} a_n = s$, and call s the *sum of the series* $\sum_{n=1}^{\infty} a_n$. If, for $\sum_{n=1}^{\infty} a_n$, the associated sequence of partial sums (s_n) does not converge, we say that $\sum_{n=1}^{\infty} a_n$ is *divergent.*

Clarification: If a series is convergent, we will indulge in a bit of an abuse of notation/terminology, and permit ourselves to say that the series $\sum_{n=1}^{\infty} a_n$ both *equals* and *converges to s.*

Directly showing the convergence or divergence of a series' (s_n) is often impractical or even impossible, but is useful in telescoping partial sums like that of (for example) $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$ which is convergent, or those of $\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right)$ which is divergent.

Theorem 8.2.6: If $\sum_{n=1}^{\infty} a_n$ is convergent then $(a_n) \to 0$.

Theorem 8.2.6 is rarely useful in its direct form (although it is handy in conjunction with the Ratio Test - to appear - for showing that $\left(\frac{n!}{n^n}\right) \to 0$). The contrapositive form is, however, very handy for showing that certain series diverge:

Theorem 8.2.7. (Test For Divergence - TFD): If $(a_n) \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

In our dealings with examples of alternating sequences which fail to converge to 0 we found two subsequences convergent to different limits. Things are slightly less complicated for using the TFD on alternating series whose terms fail to converge to 0.

Example: To show that $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$ diverges, we need only show that $\left(\frac{2n}{2n+1}\right) \to 1 \neq 0$, since then by the Subsequence Theorem, $\left((-1)^n \frac{n}{n+1}\right)$ cannot converge to 0.

Theorem 8.2.8: Let $\sum a_n$ and $\sum b_n$ be convergent series, and $c \in \mathbb{R}$. Then the series $\sum c a_n$ and $\sum (a_n \pm b_n)$ are convergent, and

$$\begin{aligned} \sum c a_n &= c \sum a_n \\ \sum (a_n \pm b_n) &= \sum a_n \pm \sum b_n \end{aligned} \tag{2}$$

The only real caveat that should accompany this theorem is that we never write

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$$

until after we've shown that $\sum a_n$ and $\sum b_n$ are convergent. So when presented with, for example, $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$, we write "Consider $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$" and go to work on that series to show it's convergent, then "Consider $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$," go to work on that series to show it's convergent, *then* invoke the conclusion of Theorem 8.2.8 to yield that the series $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$ converges (to $\frac{5}{2}$ in fact).

Comparison Test - CT: Let $\sum a_n$ and $\sum b_n$ be series with nonnegative terms. If there exists $c \in \mathbb{R}^+$ such that

1. $a_n \leq c\, b_n$ eventually and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
2. $a_n \geq c\, b_n$ eventually and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

Limit Comparison Test - LCT - Improved: Let $\sum a_n$ and $\sum b_n$ be series with nonnegative terms.

1. If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) \in [0,\infty)$ and $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

2. If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) \in (0,\infty]$ and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

It should be noted that if the CT works for series $\sum a_n$ and $\sum b_n$ then the LCT will work. (In fact, the converse of this is true as well, but often requires choosing some positive constant which complicates the problem, so we'll steer clear of this idea.)

Alternating Series Test - AST: Let $\sum a_n$ be an alternating series (where either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$, with $b_n \geq 0$). If

1. (b_n) is eventually decreasing and
2. $(b_n) \to 0$

then $\sum a_n$ converges.

Ratio Test - RT: Let $\sum a_n$ be a series, and let $\ell := \lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ if such limit exists, or $\ell := +\infty$ in the case that $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ is unbounded.

- If $\ell < 1$ $\sum a_n$ is absolutely convergent, and hence convergent.
- If $\ell > 1$ or $\ell = +\infty$ then $\sum a_n$ diverges.

Note that no assertion is made if ℓ does not satisfy either of the above cases, in particular if $\ell = 1$.

The Ratio Test is often particularly useful where factorials are present. For the purposes of concept image, note that for geometric series, $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ is a constant sequence. A reasonable interpretation of the RT is to say that if ℓ is finite, the Ratio Test measures whether or not a given series $\sum a_n$ "acts like" a geometric series as $n \to \infty$, and permits one to draw a conclusion regarding convergence of $\sum a_n$ based on the "limit" common ratio ℓ.

Power Series Representation Theorems

From this point forward in the course, you are permitted to guess correctly the limit of any numerical sequence; more accurately, you are no longer required to justify by theorem how you arrive at the value of any limit of a numerical sequence. If you're still enrolled and passing, you're entitled to this privilege.

Theorem *Power Series Substitution - PSS:* Let a be a (typically nonzero) constant. If $\sum c_n x^n$ is a PSR for $f(x)$, then a PSR for $f(ax^k)$, $k \in \mathbb{N}$ is $\sum c_n a^n x^{nk}$.

This theorem is useful for producing series to represent functions of the form $g(x) = \dfrac{1}{1 \pm x^k}$.

Theorem ***Multiplication of Power Series by a Monomial - MPSM:*** Let $\sum c_n x^n$ is a PSR for $f(x)$, and suppose that the lowest-degree monomial in said PSR is x^m for some $m \in \mathbb{N} \cup \{0\}$. Let $k \in \mathbb{Z}$ (that is, k is an integer, perhaps negative) with $m+k \geq 0$. Then a PSR for $x^k f(x)$, $k \in \mathbb{N}$ is $\sum c_n x^{n+k}$.

The extra gyrations regarding "the lowest-degree monomial..." etc., are useful for cases where we can actually *divide* a power series by a monomial, and not be left with any negative powers of x. This theorem is useful for producing series to represent functions of the form $h(x) = \dfrac{x^k}{1-x}$. Of course both Theorems MPSM and PSS can be used in conjunction. Also, note that the interval of convergence computation requires a bit of care.

Example: Consider $\dfrac{x^5}{7+x^2}$. We first "simplify" $\dfrac{1}{7+x^2} = \dfrac{\frac{1}{7}}{1+\frac{x^2}{7}}$. By Theorem PSS, we have

$$\frac{\frac{1}{7}}{1-\left(-\frac{x^2}{7}\right)} = \frac{1}{7}\sum_{n=0}^{\infty}\left(-\frac{x^2}{7}\right)^n = \frac{1}{7}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{7}\right)^n.$$

This series converges by Theorem 8.2.4 exactly when $\left|-\dfrac{x^2}{7}\right| < 1$, or when $x \in (-\sqrt{7}, \sqrt{7})$. By Theorem MPSM,

$$\frac{x^5}{7+x^2} = \frac{\frac{x^5}{7}}{1+\frac{x^2}{7}} = \frac{x^5}{7}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{7}\right)^n.$$

Has the interval of convergence changed by multiplying this series by $\dfrac{x^5}{7}$? For any given $x \in (-\sqrt{7}, \sqrt{7})$, note that we have only multiplied the series by a constant, which, by Theorem 8.2.8, does not alter the property of convergence, only the sum itself. So the IOC for $\dfrac{x^5}{7}\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{7}\right)^n$ remains $(-\sqrt{7}, \sqrt{7})$.

Multiplying any divergent series by a constant does not somehow convert it to a convergent series and vice-versa (this is again by Theorem 8.2.8), nor does this operation change the radius of convergence or the IOC. If we substitute a monomial ax^p for x in a PSR, we may alter the radius of convergence and hence the IOC, but in a predictable way. We have the following theorem:

Theorem ***(The IOC Under PSS and MPSM - IOCMPSMPSS):*** Let f be a function for which a PSR g is defined with IOC $(-R, R)$, and suppose $a \in \mathbb{R}$. Suppose further that the lowest-degree monomial in said PSR is x^m for some $m \in \mathbb{N} \cup \{0\}$, let $p \in \mathbb{N}$, and let $k \in \mathbb{Z}$ such that $pm + k \geq 0$. The IOC for $x^k g(ax^p)$ is $\left(-\sqrt[p]{\dfrac{R}{|a|}}, \sqrt[p]{\dfrac{R}{|a|}}\right)$.