Topic 9: Derivatives and Optimization

Throughout these activities, unless explicitly stated otherwise, $f(x)$ is assumed to be smooth (not pointed) and "the function $f(x)$" also means or an appropriate portion of $f(x)$ and "the derivative $f'(x)$" also means or an appropriate corresponding portion of $f'(x)$.

The Second Derivative

The second derivative of a function, $f(x)$, is simply the derivative of the derivative, $f'(x)$ and is denoted $f''(x)$ ("f double prime of x"). The second derivative is useful in a variety of settings. We already know that:

If a function graph is **increasing** then the derivative values of that function are **positive.**

If a function graph is **decreasing** then the derivative values of that function are **negative.**

If a function graph is **concave up,** the graph for the derivative of that function is **increasing.**

If a function graph is **concave down,** the graph for the derivative of that function is **decreasing.**

1. Thinking of the derivative as a function; combine these ideas and answer the following:

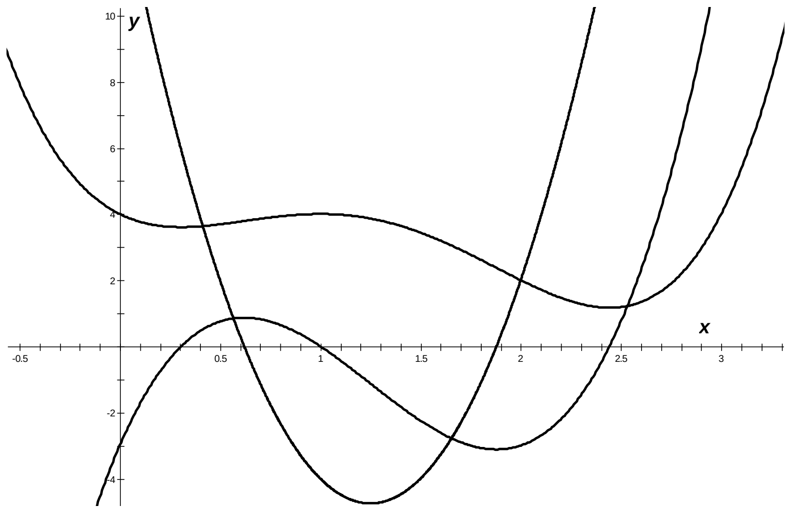
   a. When the graph of the function, $f(x)$ is concave up, the graph of the derivative is increasing and the second derivative values are: ___________.

   b. When the graph of the function, $f(x)$ is concave down, the graph of the derivative is decreasing and the second derivative values are: ___________.

   c. When the graph of the function, $f(x)$ is at a (non-endpoint) local maximum, then the function graph should be concave ________ , the derivative value should be ___________, and the second derivative value at that point should be ___________.

   d. When the graph of the function, $f(x)$ is at a (non-endpoint) local minimum, then the function graph should be concave ________ , the derivative value should be ___________, and the second derivative value at that point should be ___________.

2. The following sketch is a function and its first and second derivatives. Label the curves $f(x)$ $f'(x)$ and $f''(x)$ (you don't need to explain why for this part of the question).



Summary Table

$f(x)$	$f'(x)$	$f''(x)$
increasing & concave up		
increasing & concave down		
decreasing & concave up		
decreasing & concave down		

Definition: A **critical value** of $f(x)$ is a value, a, where $f'(a) = 0$. If the function is defined on an interval (c,d), then c and d are also considered critical values for $f(x)$.

Definition: An **extremum** for $f(x)$ is a maximum or minimum or local maximum or local minimum. (Plural: extrema or extremums)

3. Extrema occur at ___________________________,

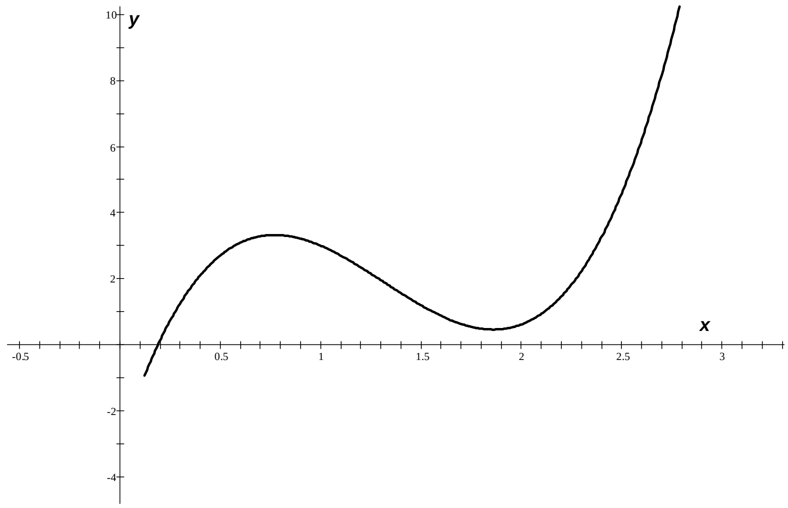
   BUT____.

4. If $x = a$ is an extremum for $f(x)$, then $f(a)$ is a local maximum if ________________.

5. If $x = a$ is an extremum for $f(x)$, then $f(a)$ is a local minimum if ________________.

6. If $x = a$ is an extremum for $f(x)$, and $f''(x) = 0$ then…

An ***inflection point*** of a curve is a point where the concavity changes (from up to down or down to up). Identify the inflection point on the curve below.



7. If $f(x)$ has an inflection point at $(a, f(a))$ where the concavity changes from down to up, then for x to the "left" of a the second derivative, $f''(x)$ is _____________; and for x to the "right" of a the second derivative, $f''(x)$, is _____________; thus $f''(a)$ is _____.

8. Similarly, if $f(x)$ has an inflection point at $(a, f(a))$ where the concavity changes from up to down, then for x to the "left" of a the second derivative, $f''(x)$ is _____________; and for x to the "right" of a the second derivative, $f''(x)$, is _____________; thus $f''(a)$ is _____.

9. Consider the function $f(x) = 2x^3 - 2x^2 - 12$. (Domain is all real numbers so no endpoints)

   a. Find all critical values for f(x).

   b. For each critical value determine if it is a local max, local min, or neither.

   c. Determine if f(x) has any inflection points.

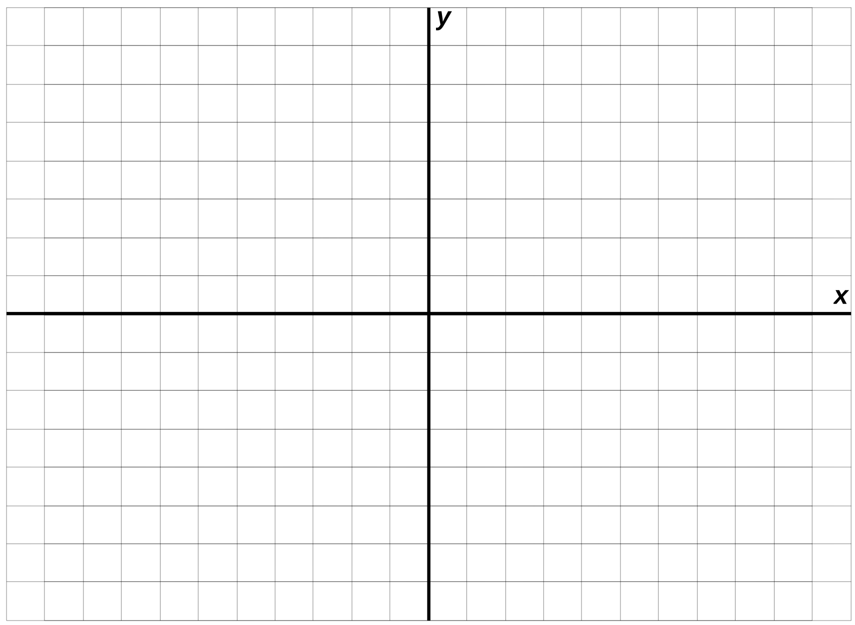
10. The function $f(x) = 2x^3 - 9x^2 - 60x + 50$ is defined on all of the real numbers and has one local maximum value and one local minimum value.

a. Find the local max and min of $f(x)$.

b. Determine any inflection points for $f(x)$.

c. Where is $f(x)$ concave up and where is $f(x)$ concave down?

d. Sketch the best possible graph you can for $f(x)$ (without graphing it on your calculator). Use the information you determined in parts a), b) and c), and any needed function values from your calculator. Label clearly.



11. Suppose a rock is thrown directly upward, and at time *t* seconds after it is released, the height of the rock is given by the function $h(t) = -16t^2 + 64t$ feet above ground (note the 16 come from the gravitational constant of planet Earth; -32 feet /second^2).

    a. What is the initial height of the rock? (Think about what this means in terms of time.)

    b. When does the rock hit the ground? (Think about what this means in terms of height.) Determine this algebraically, show your work.

    c. What is the average speed of the rock during the rock's first second of flight? Determine this algebraically, show your work.

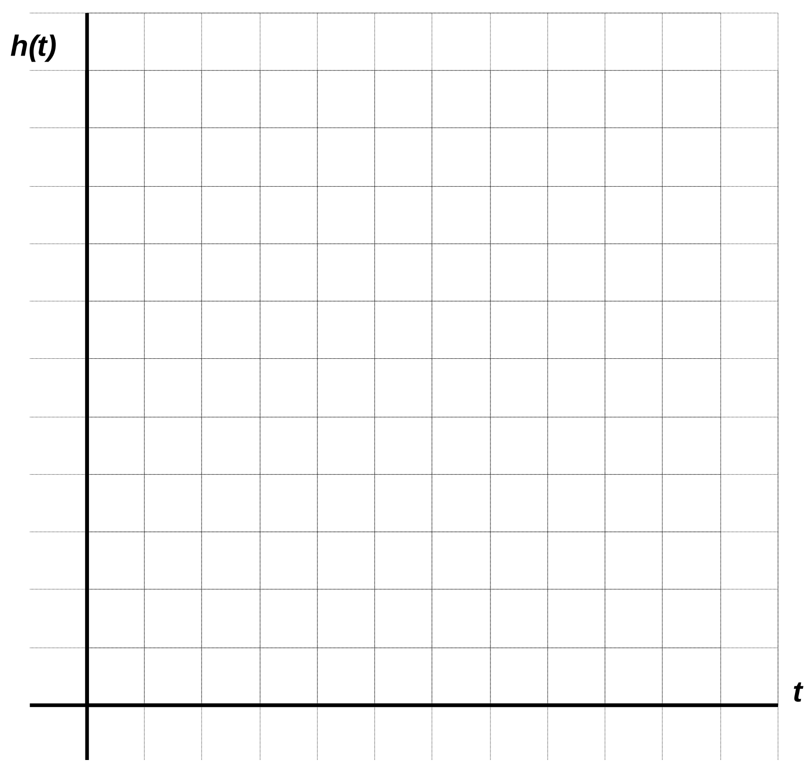
    In this setting, $h'(t)$ is the <u>velocity</u> (speed with direction in feet / second) of the rock.

    d. What is the speed of the rock when $t = 1$ second and is the rock going up or down? How do you know? Use the first derivative to determine this value algebraically, show your work; include units. At what other time is the rock going the same speed and is the rock going up or down then? Determine this algebraically, show your work.

e. What is the initial velocity of the rock?

f. When is the rock at its highest point and how high does it go? Use calculus and determine this algebraically; show your work.

g. Sketch the height function of the rock, mark all of the points from the preceding questions on the graph. Don't forget to label the axes with units for the **height** function.



12. Suppose you wish to find a pair of positive whole numbers, a and b, whose sum is 100 and whose product is as large as possible. Find the solution to this question using calculus.