Topic 12: The Chain Rule

Suppose you wished to differentiate a function such as $f(x) = (2x-1)^7$. Even with Pascal's Triangle and the Binomial Coefficients, this is somewhat cumbersome to multiply out. Using the Product Rule repeatedly would lead to tedious algebra. For a function such as $g(x) = \sqrt{2x-1} = (2x-1)^{\frac{1}{2}}$, neither of these approaches would work. Before we discover a new rule, *the Chain Rule*, to differentiate such functions, we will review function composition.

Function Composition Review

Both $y = (2x-1)^7$ and $y = \sqrt{2x-1} = (2x-1)^{\frac{1}{2}}$ can be thought of as *functions of a function*. For a function $f(x)$, to evaluate the function at $x=1$ or $x=a$, you would simply replace every instance of x in the function rule with the corresponding value ($x=1$ or $x=a$). To create a *function of a function*, you replace every instance of x in the function rule with another function rule as shown in the following example.

Example: Writing $y = (2x-1)^7$ as Function Composition

Consider the functions $F(x) = 2x-1$ and $G(x) = x^7$. The function $y = (2x-1)^7$ reads "$2x-1$ to the seventh power." This naturally leads to the idea of the function $2x-1$ on the INSIDE of the function "to the seventh power" where "to the seventh power" is the OUTSIDE function. Symbolically this means:

$$G(2x-1) = (2x-1)^7$$

Since $F(x) = 2x-1$, this is also written as $G(F(x)) = G(2x-1) = (2x-1)^7$. We say $y = (2x-1)^7$ is G of F of x for $G(x) = x^7$ and $F(x) = 2x-1$.

1. For $F(x) = x^3$ and $G(x) = x+1$, determine $G(F(x))$ and determine $F(G(x))$.

2. For $F(x) = 2x^3$ and $G(x) = 3x+1$, determine $G(F(x))$ and determine $F(G(x))$.

4. For $F(x) = \cos(x)$ and $G(x) = x^3 + 2x + 1$, determine $G(F(x))$ and determine $F(G(x))$.

5. For $F(x) = \sin(x)$ and $G(x) = e^x$, determine $G(F(x))$ and determine $F(G(x))$.

6. Write $y = \sqrt{2x-1}$ as $G(F(x))$ by first writing the inside function as $F(x)$ and the outside function as $G(x)$.

7. Write $y = (3x^2 + 4)^{11}$ as $G(F(x))$ by first writing the inside function as $F(x)$ and the outside function as $G(x)$.

8. Write $y = \sin^4(x)$ as $G(F(x))$ by first writing the inside function as $F(x)$ and the outside function as $G(x)$. (Note $\sin^4(x) = (\sin(x))^4$. The first way to write it is more standard.)

9. Write $y = e^{2x^4+3}$ as $G(F(x))$ by first writing the inside function as $F(x)$ and the outside function as $G(x)$.

Deriving the Chain Rule for $f(x)^n$

Note: $f(x)^n$ is a composed function where $f(x)$ is the inside function and $y = x^n$ is the outside function.

10. Use the Product Rule to differentiate $f(x)^2$. Simplify your solution completely.

11. Use the Product Rule to differentiate $f(x)^3$. Start by thinking of $f(x)^3$ as the product of two functions $f(x)^2 \times f(x)$ and use your results from the previous question for $[f(x)^2]'$. Simplify your solution completely.

12. Use the Product Rule to differentiate $f(x)^4$. Start by thinking of $f(x)^4$ as the product of two functions $f(x)^2 \times f(x)^2$ and use your results from the question 5 for $[f(x)^2]'$. Simplify your solution completely.

13. What is your best guess for the derivative of $f(x)^n$?

In general, the derivative of a function to a power, a special case of the <u>Chain Rule</u>, is:

$$[f(x)^n]' = nf(x)^{n-1} f'(x)$$

14.

a. Multiply out $y = (2x^3 + 5)^2$ completely and then determine y' using the Sum, Power and Constant Multiples Rules.

b. Use the Chain Rule to determine the derivative of $y = (2x^3 + 5)^2$.

15. Use the Chain Rule to determine the derivative of $y = (2x - 1)^7$.

16. Use the Chain Rule to determine the derivative of $y = \sqrt{2x - 1} = (2x - 1)^{\frac{1}{2}}$.

17. Use the Chain Rule to determine the derivative of $y = (3x^2 + 4)^{11}$.

18. Use the Chain Rule to determine the derivative of $y = \dfrac{1}{(3x^2 + 4)^{11}}$.

The Chain Rule: The chain rule applies more generally to the composition of any two functions (not just functions involving polynomials). If you have an "inside" function $g(x)$ and an "outside" function $f(x)$ then the derivative of the composition $f(g(x))$ is

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

People often think of it verbally this way : "The derivative of the outside function evaluated at the inside function times the derivative of the inside function."

Example:

Suppose we want to find the derivative of $y = \sin(x^2 - 3x)$
First we identify the "outside" function $f(x)$ and "inside function" $g(x)$

$$f(x) = \sin(x) \text{ and } g(x) = x^2 - 3x. \text{ Then}$$
$$f'(x) = \cos(x) \text{ and } g'(x) = 2x - 3$$

So the chain rule says $y' = f'(g(x)) \cdot g'(x)$, plugging in we get

$$y' = \cos(x^2 - 3x) \cdot (2x - 3)$$

Use the chain rule to find the following derivatives:

19. $y = e^{2x^4+3}$

20. $y = \sin^4(x)$

21. $y = \dfrac{1}{\cos(x)} = (\cos(x))^{-1}$