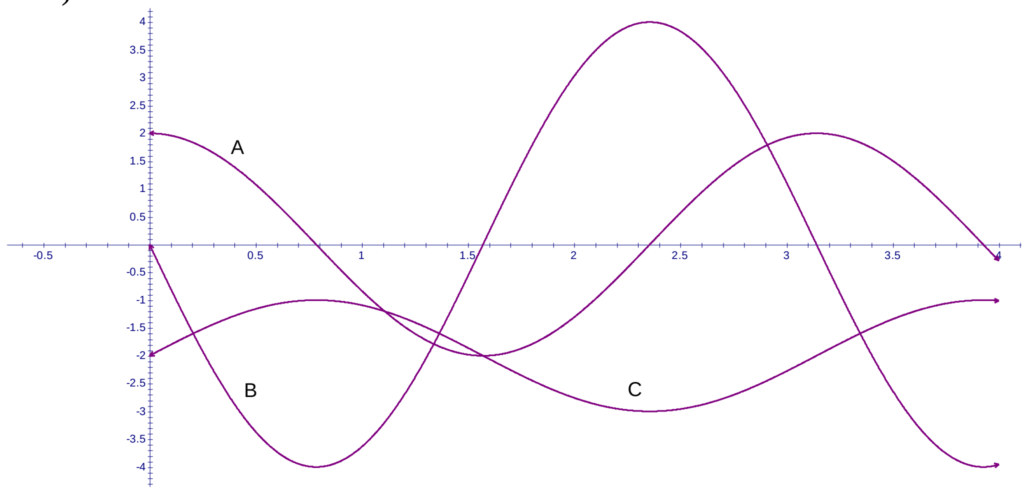
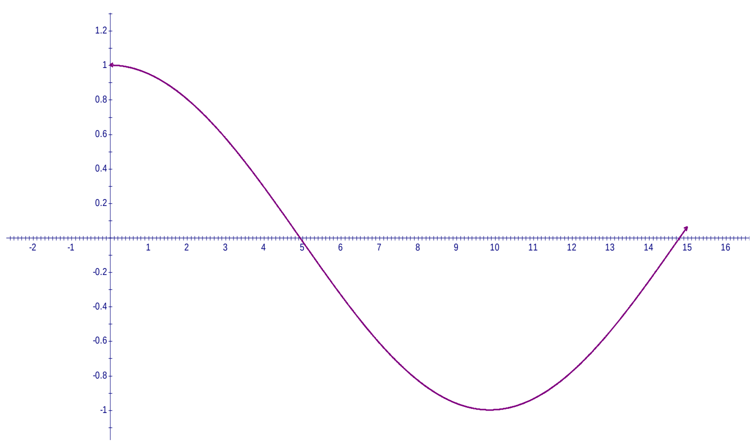
- The Final Exam is Wednesday 6/13 @ 4-5:50 PM in our usual room (MNB 104)
- You may have 2 notecards, calculator and a ruler (rulers and calculators can be borrowed during the exam)
- In addition to reviewing the coursepack and homework questions for the term, the following questions are for review. The answers will be posted on our course webpage sometime over the weekend.

1. The graphs A, B and C are a function and its first and second derivatives (not necessarily in that order). Which is which?



ANSWER: The function is C; The first derivative is A; The second derivative is B

2. Suppose that the graph below represents the *derivative*, $f'(x)$, of some function, $f(x)$. The axes are a little hard to read, but assume the y-intercept is (0,1) and the x-intercepts are (5,0) and (14.75,0). For each question below answer the question or explain why the answer cannot be determined from the graph of $f'(x)$.



   a. On what interval(s) is $f(x)$ increasing? ANSWER: $0 \leq X \leq 5$ (where the derivative is positive)

b. On what interval(s) is $f(x)$ concave down? ANSWER: $0 \leq X \leq 10$ (where the derivative is decreasing)
c. On what interval(s) is $f(x)$ negative? ANSWER: You can't tell from the graph. The derivative just tells the shape of the graph so you can't tell whether the actual function values are positive or negative just from looking at the derivative.
d. Where does $f(x)$ have an inflection point? ANSWER: At X=10 (where the concavity changes in f(x) – which corresponds to where the derivative changes from decreasing to increasing – local mins or maxs of the derivative correspond to inflection points on f(x))
e. Where is $f''(x) > 0$? ANSWER: $X \geq 10$ (where the derivative is increasing)
f. The point $(5,0)$ is a critical point for $f(x)$ (since a critical point is where the derivative is zero). Is $(5,0)$ a local max, min, or neither? Explain. ANSWER: (5,0) is a local max since the derivative is decreasing at that point so the function is concave down)

3. Suppose you are constructing a rectangular fence to enclose 100 square feet of garden. One side of the fence must be wood and costs \$20 per linear foot to construct; the other three sides are wire and cost \$5 per linear foot to construct. Additionally, the side parallel to the wooden side will have shrubs planted along it at a cost of \$10 per linear foot. Use calculus to determine the dimensions and cost (including the shrubs) of the least cost fence.

   ANSWER: Let L be the length of the wooden side of the fence. Then the cost function (including the cost of the shrubs) is given by:
$$C(L) = 35L + \frac{1000}{L} = 35L + 1000L^{-1}$$
   To find the minimum cost we look for critical points (where the derivative is zero):
$$C'(L) = 35 - \frac{1000}{L^2} = 0$$
   Solving for L we get $L \sim 5.35\ ft$. Let's check that this is a minimum. If we look at the second derivative at this point we get:
$$C''(5.35) = \frac{2000}{(5.35)^3} > 0$$
   Since this is positive we see the graph is concave up so we have a minimum. We get that our dimensions are:
   Length of wooden side 5.35 feet, width 18.69 ft
   Least cost: \$374.17

4. Suppose a rock is hurled vertically up (on Earth, where the acceleration due to gravity is -32 ft/sec^2). Suppose further that the rock is 186 feet above ground at $t = 2$ seconds and the velocity at $t = 3$ seconds is 4 feet / second.

   (i) Find the height ($h(t)$), velocity ($v(t)$) and acceleration ($a(t)$) functions)
   ANSWER: $a(t) = -32\ ft/sec^2$;
$$v(t) = \int a(t)dt = \int -32dt = -32t + c$$

We are given that the velocity at t=3 is 4 ft/sec so we can plug this in to solve for c:

$$v(3) = -32(3) + c = 4$$

Solving gives c=100 so $v(t) = -32t + 100\ ft/sec$

Finally,

$$h(t) = \int v(t)dt = \int -32t + 100\ dt = \frac{-32}{2}t^2 + 100t + C = -16t^2 + 100t + C$$

We are given that the rock is 186 feet above ground at t=2 seconds so we can plug this in to solve for C:

$$h(2) = -16(2)^2 + 100(2) + C = 186$$

Solving for C gives C=50 so $h(t) = -16t^2 + 100t + 50\ ft$

In summary:

$$a(t) = -32\ ft/sec^2$$
$$v(t) = -32t + 100\ ft/sec$$
$$h(t) = -16t^2 + 100t + 50\ ft$$

(ii) Completely describe the path of the rock (max height, when did it hit the ground, how fast was it going when it hit the ground)

ANSWER:

To find the max height we should find where the first derivative is zero (this is the critical point).

$$h'(t) = v(t) = -32t + 100 = 0.$$

Solving for t gives $t = 3.125$. Since the second derivative, a(t) is always negative we know that the graph of h(t) is concave down so we have a local max so to find the max height we plug this time in to our height function:

$$h(3.125) = -16(3.125)^2 + 100(3.125) + 50 = 206.25\ ft.$$

The rock hit the ground when it's height was zero:

$$h(t) = -16t^2 + 100t + 50 = 0$$

Using the quadratic formula to solve for t we get t=6.7 seconds (the other value is negative so we don't use it). To find out how fast it was going when it hit the ground we plug that time in to the velocity function:

$$v(6.7) = -32(6.7) + 100 = -114.4\ ft/sec$$

We could also figure out the initial velocity and height by plugging in 0 to the velocity and height functions:

$$v(0) = 100\frac{ft}{sec}; h(0) = 50\ ft$$

In summary:

The rock was thrown into the air at an initial height of 50 ft with an initial velocity of 100 ft/sec. The rock reached a maximum height of 206.25 ft after 3.125 seconds. The ball hit the ground after 6.7 seconds and was going 114.4 ft/seconds when it hit the ground.

(iii) Determine the speed of the rock at $t = 1.5$ seconds (find the equation for and graph the corresponding tangent line) and find another time when the rock is going the same speed it was going at $t = 1.5$ seconds (find the equation for and graph the corresponding tangent line).

Answer: Determine the speed of the rock at t=1.5 seconds : At 1.5 seconds the graph was going

$$v(1.5) = -32(1.5) + 100 = 52\ feet/sec.$$

Find the equation for and graph the corresponding tangent line: This (52) is the slope of the tangent line at 1.5. To find the equation of the tangent line we also need to know that $h(1.5) = 164$. So we are looking for the tangent line at (1.5, 164) and the slope is 52:

y=52x+ b (plug in (1.5, 164) to solve for b and get

$$y = 52x + 86$$

Find another time when the rock is going the same speed it was going at t=1.5 seconds: At 1.5 seconds the velocity was 52 feet/sec so we note the rock was going up. It will also travel 52 feet/sec again on its way back down. So to find that time we just set up this equation and solve for t:

$$v(t) = -32t + 100 = -52$$

Solving for t gives

$$t = 4.75\ seconds$$

Find the equation for and graph the corresponding tangent line: Now we have a slope of -52 and our point is t=4.75, h(4.75)=164. Using the same process as before we find:

$$y = -52x + 411$$

(iv) Find the average speed of the rock over the first three seconds of travel (what calculus concept does this relate to?).

Answer: This corresponds to the slope of the secant line through (0,h(0)) and (3,h(3)).

$h(0) = 50, h(3) = 206$. We want the slope through (0,50) and (3,206):

$$\frac{206-50}{3-0} = \frac{156}{3} = 52\ ft/sec$$

5. Find the derivative of each of the following

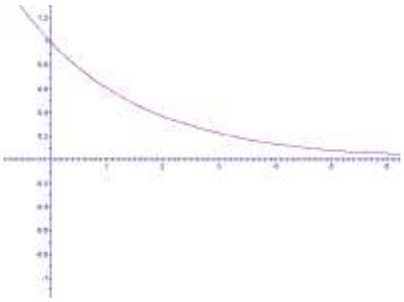
a. $sin(x^3 - 3x)$ b. $y = (1+7x)(3x+4)^{13}$ c. $y = (1+7x^3)^{12}$ d. $y = \frac{1+7x^3}{2-x^4}$

Answer:

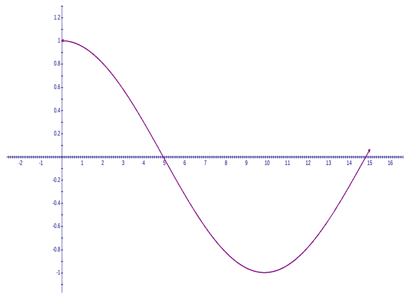
$$a.\ \cos(x^3 - 3x)\,(3x^2 - 3)\ (chain\ rule)$$
$$b.\ 7(3x+4)^{13} + 39(1+7x)(3x+4)^{12}\ (product\ rule)$$
$$c.\ 252x^2(1+7x^3)^{11}\ \ (chain\ rule)$$
$$d.\ \frac{21x^2(2-x^4) + 4x^3(1+7x^3)}{(2-x^4)^2}\ (quotient\ rule)$$

6. For each of the following graphs, i) resketch the graph on graph paper, assume the graph is the function and sketch the derivative, ii) resketch the graph on graph paper, assume the graph is the derivative and sketch an antiderivative

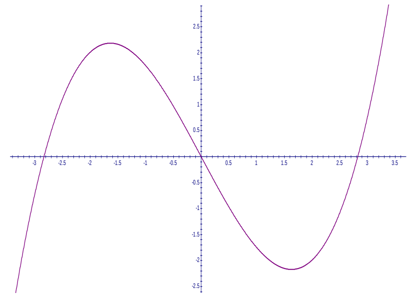
a.



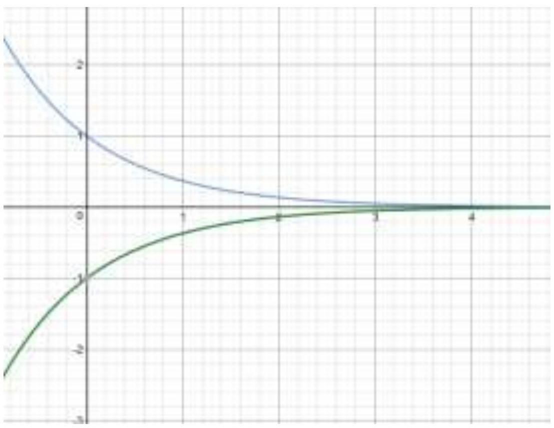
b.



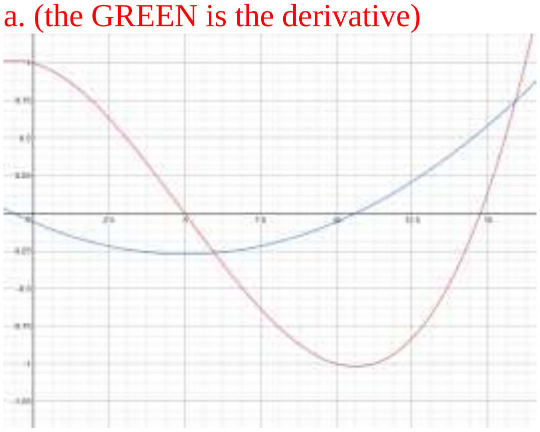
c.



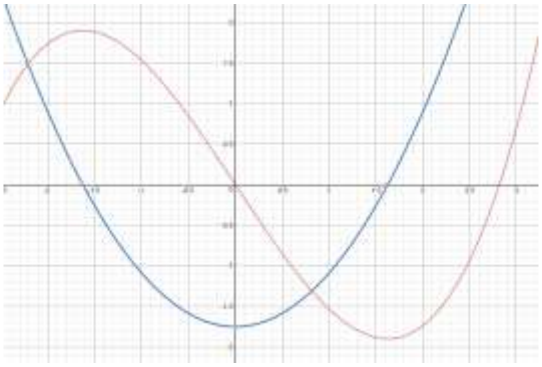
ANSWER: Part i) sketch the derivative



a. (the GREEN is the derivative)

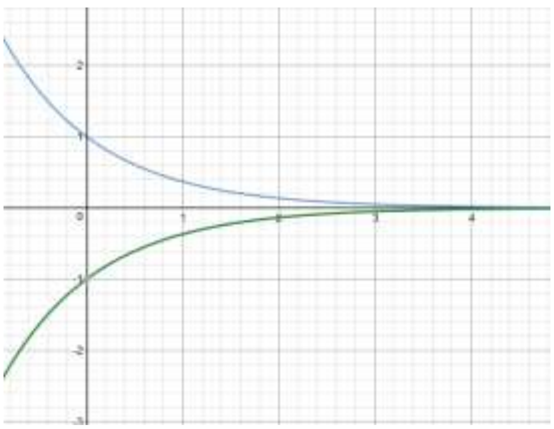


b. (the BLUE is the derivative)

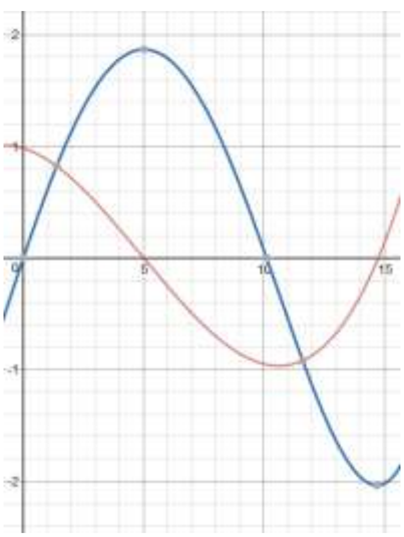


c. (the blue is the derivative)

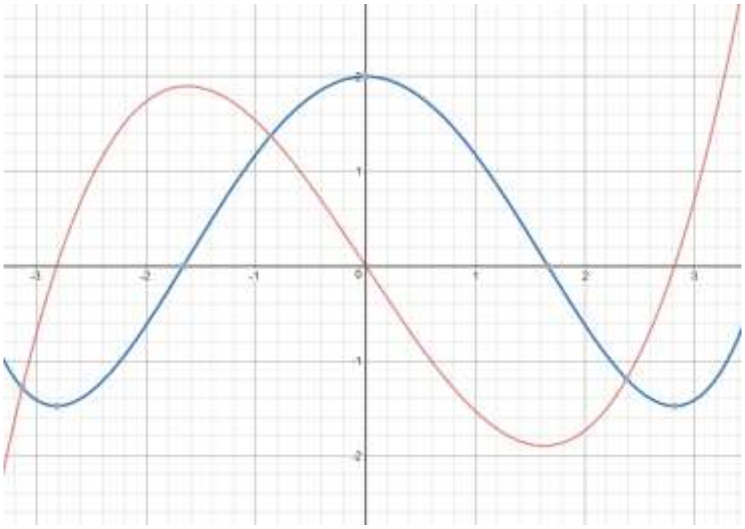
Part ii) sketch an antiderivative



a. The GREEN is the antiderivative (it just looks the same as the derivative)



b. (the blue is an antiderivative)



c. (the blue is an antiderivative)

7. Using calculus, compute each of the following:

a. $\int_{-2}^{4} 2x^3 + x \; dx$ b. $\int_{1}^{4} \frac{2}{x^3} \; dx$ c. $\int 2\cos(x) - e^x dx$

Answer:

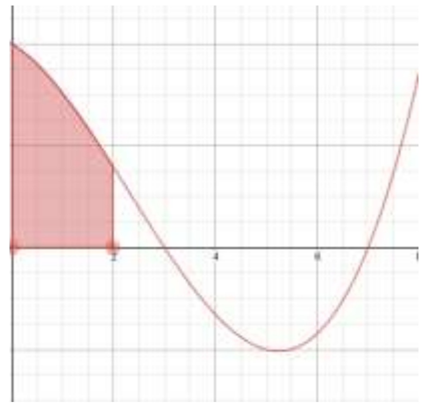
$$a. \int_{-2}^{4} 2x^3 + x \; dx = \frac{2}{4}x^4 + \frac{1}{2}x^2 |_{-2}^{4} = \left(\frac{2}{4}(4)^4 + \frac{1}{2}(4)^2\right) - \left(\frac{2}{4}(-2)^4 + \frac{1}{2}(-2)^2\right)$$

$$= 136 - 10 = 126$$

$$b. \int_{1}^{4} \frac{2}{x^3} \; dx = \int_{1}^{4} 2x^{-3} dx = \frac{2}{-2}x^{-2} = \frac{-1}{x^2} |_1^4 \left(\frac{-1}{4^2}\right) - \left(\frac{-1}{1^2}\right) = \frac{-1}{16} + 1 = \frac{15}{16}$$

$$c. \int 2\cos(x) - e^x dx = 2\sin(x) - e^x + C$$

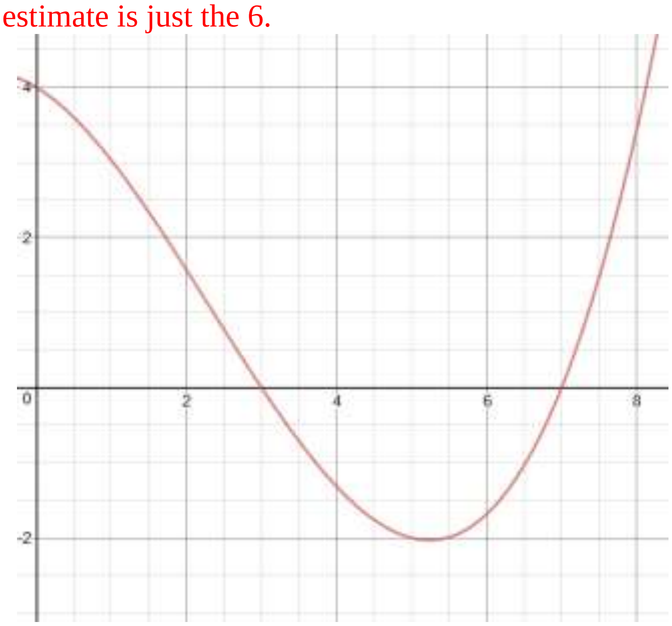
8. Write an integral that represents the indicated signed area under the graph of $f(x)$ below.



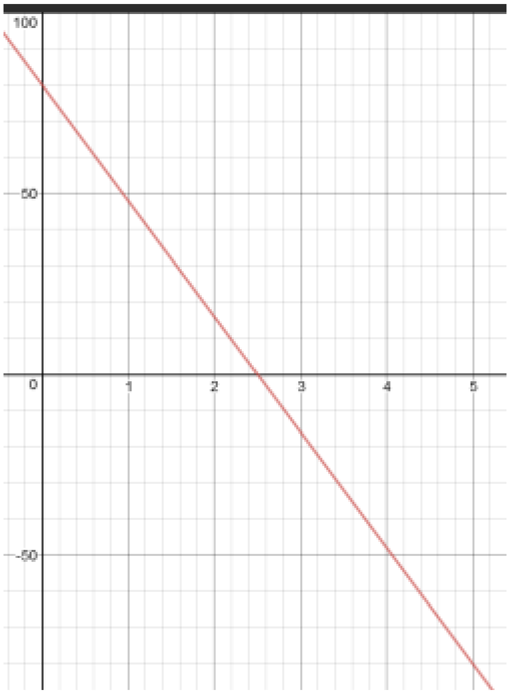
ANSWER: $\int_0^2 f(x)dx$

9. Estimate the value of $\int_0^4 f(x)dx$.

ANSWER: The area under the curve and above the axis from 0 to 2 is about 6 since the area contains almost one and a half boxes. Each box is 2x2 = 4 so about 1.5x4 = 6. The area under the curve and above the axis between 2 and 3 is about the same as the area above the curve and below the axis between 3 and 4. Since the definite integral is signed area then these values just cancel out so the area estimate is just the 6.



10. A rock is thrown into the air and the *velocity*, $v(t)$, of the rock is shown in the graph below.



a. What was the initial velocity of the rock?
Answer: We can just read this off of the velocity graph. At t=0 the y-value is 80 ft/sec. (I forgot to give units, but let's just say it is feet per second).

b. At what time did the rock reach its highest point? How can you tell from the graph?

Answer: This would be when the graph crosses the x-axis. This is the point at which the velocity changes from positive to negative. That corresponds to when the position changes from increasing (going up) to decreasing (going down). So the highest point when happen at 2.5 seconds.

c. If the rock hit the ground after 5 seconds, then write a definite integral that represents the *total* distance the rock traveled.
We can tell from the graph that the equation of the velocity function is

$$v(t) = -32t + 80$$

Answer: $\int_0^{2.5} -32t + 80dt - \int_{2.5}^{5} -32t + 80dt$

11. For each of the given (input, output) relationships in Tables One and Two; answer the following: Is the relationship a function? Why or why not?

a. A a function of B b. B a function of A
c. C a function of D d. D a function of C

For the relationships that are functions; also answer:
i. What is the domain of the function? ii. What is the range of the function?

Table One		Table Two	
A	**B**	**C**	**D**
0	2	1	3
1	3	2	6
2	3	3	9
3	4	4	12

ANSWER:
a. NO, an input of 3 has two different outputs
b. YES; Domain {0,1,2,3}, Range {2,3,4}
c. YES: Domain {3,6,9,12}, Range {1,2,3,4}
d. YES: Domain { 1,2,3,4}, Range {3,6,9,12}

12. What is the domain and range for each of the following functions? Show your work for determining the domain when applicable.

a. $f(x)=\dfrac{1}{3x-3}$ b. $f(x)=\dfrac{1}{1+x^2}$ c. $f(x)=\sqrt{5x-10}$

d. $f(x)=\dfrac{1}{\sqrt{4-x^2}}$ e. $f(x)=\dfrac{1}{\sqrt{4x-2}}$ f. $y=\sqrt{(x-2)^3}$

ANSWER:

$$a. Domain: x \neq 1, Range: y \neq 0$$
$$b. Domain: All\ real\ x, Range: 0 < y \leq 1$$
$$c. Domain\ x \geq 2. Range\ y \geq 0$$
$$d.\ \ Domain:\ -2 < x < 2.\ \ Range: y \geq 1/2$$
$$e. Domain: x > \frac{1}{2}, Range: y > 0$$
$$f. Domain: x \geq 2, Range\ y \geq 0$$

13. Use the provided graph of Eugene's bike trip to answer the following questions: Be sure you can find equations of any of the requested lines.
a. When is Eugene taking a break? How do you know?
Answer: 30-50 minutes. The graph is flat so his distance from home is not changing.
b. When is Eugene going away from or towards his house? How do you know?

Answer: He is going away from 0-13, 70-80, 90-120, and towards from 80-90. He is going away when the graph is increasing and going towards when the graph is decreasing.

c. When is Eugene's speed increasing? Explain how you know using calculus concepts.
Answer: His speed is increasing when the slope of the graph is getting steeper.

d. When is Eugene going at a constant rate and what are his speeds in mph during those times? Explain how you know using calculus concepts.
Answer: Eugene is going at a constant rate when the graph is a straight line. During those times the slope which represents his speed (instantaneous rate of change) will be constant. He is going at a constant rate from 0-30 minutes (10 mph) and 50-70 minutes (15 mph) and 100-120 minutes (24 mph)

e. When is Eugene going the fastest and how fast, in mph is he going? Which calculus concept does this relate to?
Answer: He is going the fastest just before 90 minutes. You can tell because that is when the slope of the graph (the slope of the tangent line, or the derivative) is the steepest. If you estimate the slope there you get about 92.3 mph (you get negative sign because he is going toward the house which for our graph is the negative direction)

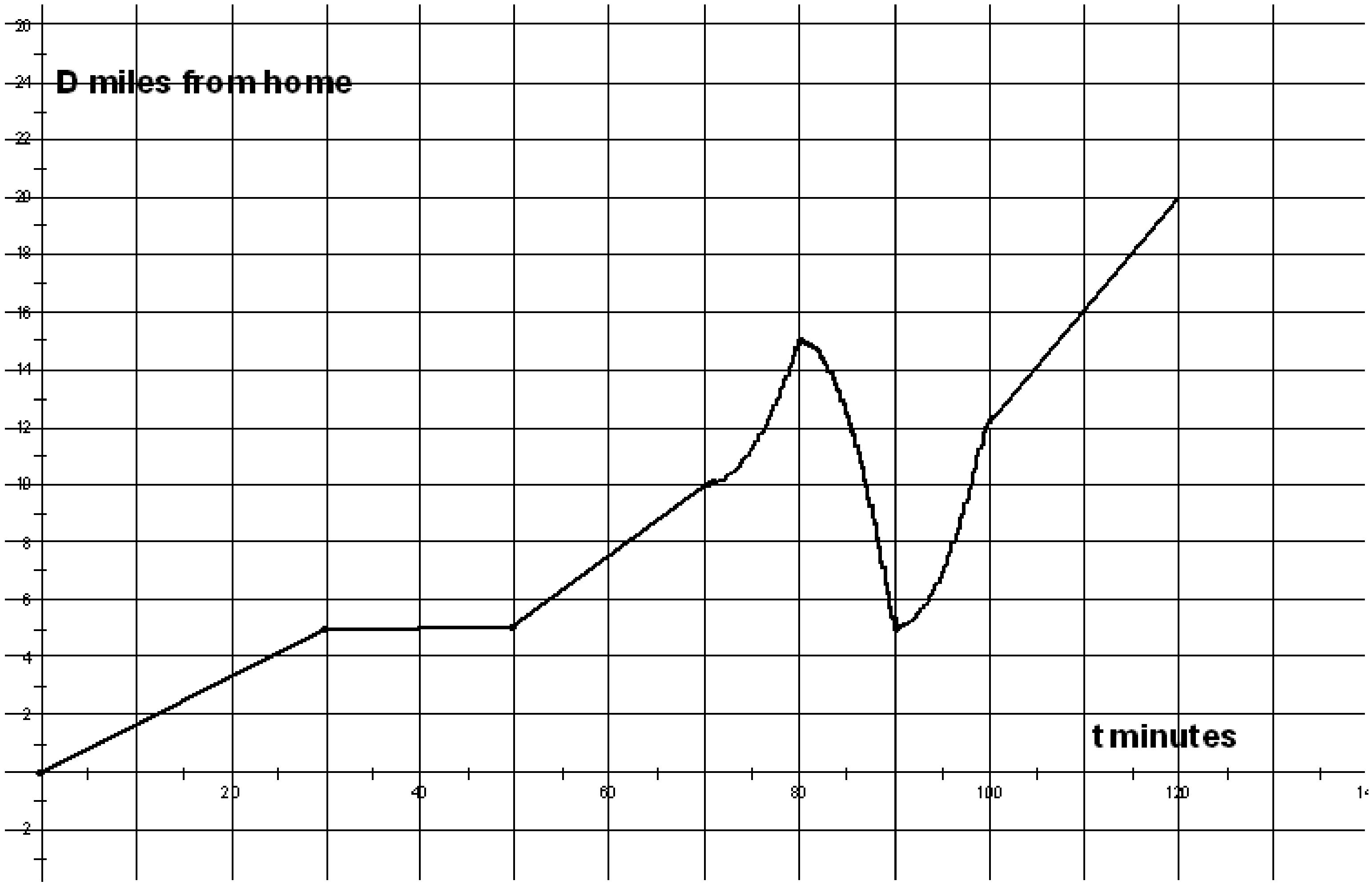
f. What is Eugene's average speed, in mph, during the first 45 minutes of his trip? The second 45 minutes? Which calculus concept does this relate to?
Answer: His average speed is 6 2/3 mph. This relates to the slope of the secant line.

g. When is Eugene's average speed 9 mph, 12 mph, 15 mph? Which calculus concept does this relate to?
Answer: Since the graph is in miles/minute we should first convert the speeds from mph to miles/minute. 9 miles/hour * 1 hour/ 60 minutes = 9/60=3/20 miles/ minute. So since this is average speed we are looking for two points where the secant line connecting them has a slope of 3/20. One such pair would be approximately (86,11) and (106, 14) .
The solutions for 12, and 15 follow similar steps.

h. When is Eugene's (instantaneous) speed 9 mph, 12 mph, 15 mph? Which calculus concept does this relate to?

Answer: The calculus concept here is the derivative or slope of the tangent line. Like in the last problem we need to convert the speeds to miles/minute. 9mph = 3/20 miles/minute. Looking on the graph it looks like the point (72,10.2) would have a tangent line with that slope. The other problems are done similarly.

MATH 495.595 / PRACTICE MIDTERM EUGENE'S BIKE TRIP



Practice Final Exam Questions

14. Use the numerical method (tables with right and left hand limits) to find the derivative of $f(x) = \cos(x)$ at $x = 1.5$. Find answer to nearest thousandth. Be sure your calculator is in radians.

Left hand limit (h<0)				Right hand limit (h>0)			
h	$f(1.5)$	$f(1.5+h)$	$\frac{f(1.5+h)-f(1.5)}{h}$	h	$f(1.5)$	$f(1.5+h)$	$\frac{f(1.5+h)-f(1.5)}{h}$
-.1	0.0707	.1699	-.9923	.1	0.0707	-.0291	-.9994
-.01	0.0707	.0807	-.9971	.01	0.0707	.0607	-.9978
-.001	0.0707	.0717	-.9975	.001	0.0707	.0697	-.9975
-.0001	0.0707	.0708	-.9975	.0001	0.0707	.07064	-.9975
Left hand limit			-.9975	Right hand limit			-.9975
$f'(1.5) = -.9975$							

15. Use Fermat's Method to compute the derivatives of $f(x) = 5x^2 + 1$ and $f(x) = 2 - x^3$. Carefully show all of your steps.

$[For\ f(x) = 5x^2 + 1]$:

$$= f'(x) = \lim_{h\to 0}\frac{5(x+h)^2 + 1 - (5x^2+1)}{h} = \lim_{h\to 0}\frac{5(x^2+2xh+h^2) + 1 - (5x^2+1)}{h}$$

$$= \lim_{h\to 0}\frac{5x^2 + 10xh + 5h^2 + 1 - 5x^2 - 1}{h} = \lim_{h\to 0}\frac{10xh + 5h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(10x+5h)}{h} = \lim_{h\to 0} 10x + 5h = 10x$$

$[For\ f(x) = 2 - x^3]$:

$$f'^{(x)} = \lim_{h\to 0}\frac{2 - (x+h)^3 - (2-x^3)}{h} = \lim_{h\to 0}\frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h}$$

$$= \lim_{h\to 0}\frac{2 - x^3 - 3x^2h - 3xh^2 - h^3 - 2 + x^3}{h} = \lim_{h\to 0}\frac{-3x^2h - 3xh^2 - h^3}{h}$$

$$= \lim_{h\to 0}\frac{h(-3x^2 - 3xh - h^2)}{h} = \lim_{h\to 0} -3x^2 - 3xh - h^2 = -3x^2$$

Practice Final Exam Questions

16. For each of the following, use the Power, Constant Multiple and Sum Rules (as appropriate) to determine $f'(x)$. Carefully show your work, don't skip steps. Give your answers without negative exponents.

a. $f(x)=\dfrac{1}{2x^3}$ b. $f(x)=\sqrt[3]{x^2}$ c. $f(x)=17x^5-22x^3+\pi$

$a.\, f'(x)=\dfrac{-3}{2x^4}$ $b.\ f'(x)=\dfrac{2}{3\sqrt[3]{x}}$ $c.\, f'(x)=85x^4-66x^2$

17. Find the equation of the tangent line to $f(x)=3x^2-5x+1$ at x=2.

We need the slope which is the derivative:

$$f'(x)=6x-5$$

Now just plug in 2:

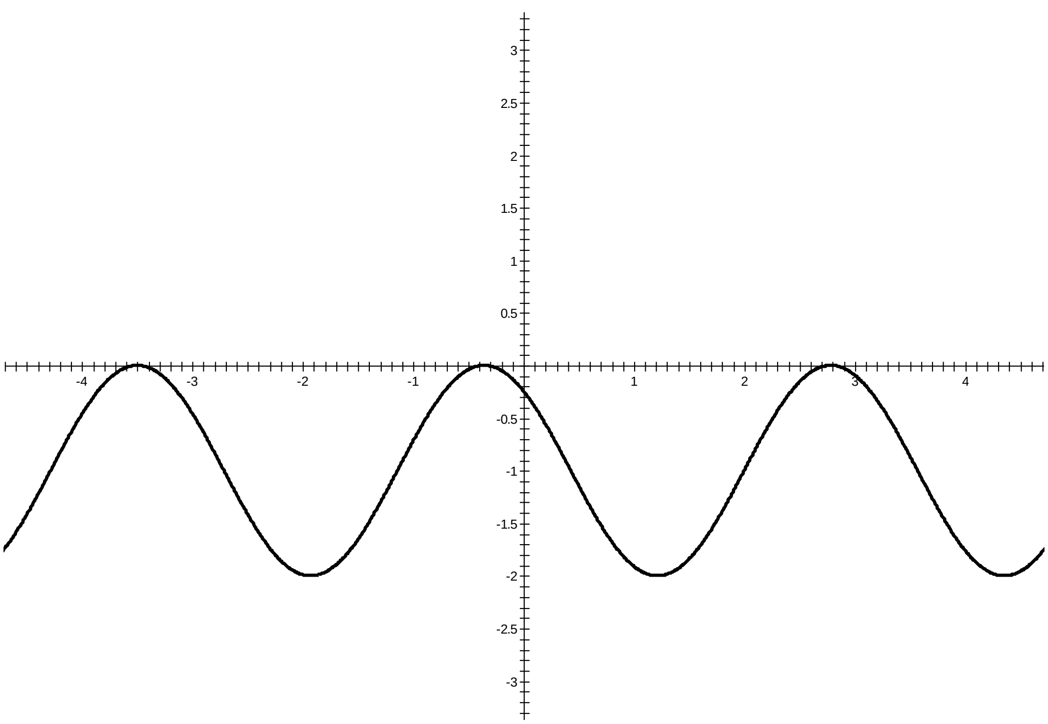
$$f'(x)=6(2)-5=7$$

So the slope is 7. The y-value is f(2)=3. So the equation is y=7x+b at (2,3). Solving gives

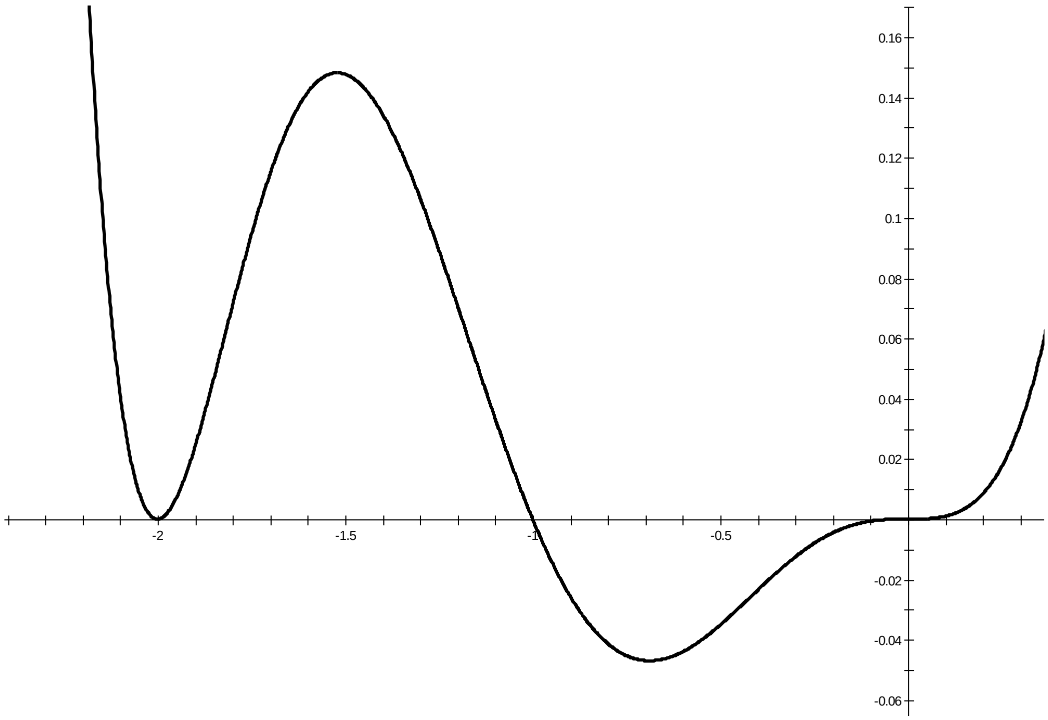
$$y=7x-11$$

18. For each of the following functions, sketch the graph of the derivative on the same axes. Carefully mark all key points on the graphs and/or axes and explain your work. Use key terms in your explanations. Label your curves $f(x)$ and $f'(x)$

Key Terms		
	Function	↑, ↓, local max, local min, concave up, concave down
	Derivative	> 0, < 0, = 0, ↑, ↓



Practice Final Exam Questions



(solutions drawing in by hand)

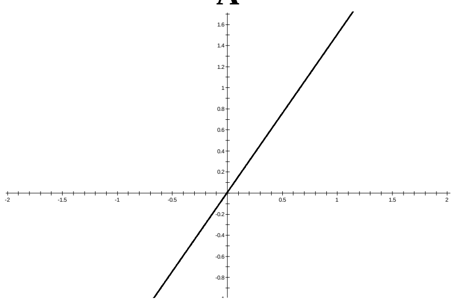
Practice Final Exam Questions

Practice Final Exam Questions

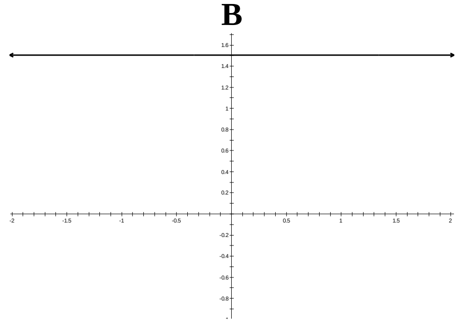
Identify as many relationships (graph A is the derivative of graph ______ and the function for graph ___, etc.) as possible. Explain your responses using key terms.

Key Terms	**Function**	↑ , ↓, local max, local min, concave up, concave down
	Derivative	> 0, < 0, $= 0$, ↑ , ↓

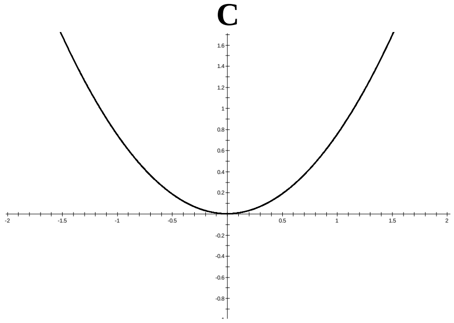
A



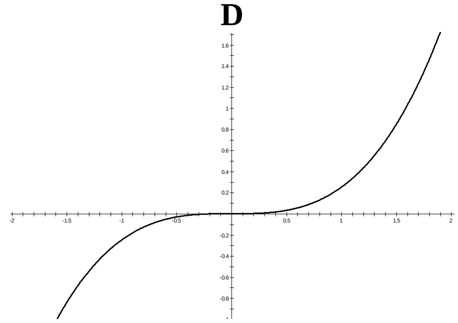
B



C



D



B is the derivative of A
A is the derivative of C
C is the derivative of D
You should add in key terms to explain
You might also argue that D is the derivative of C

19. Consider $f(x) = \frac{1}{3}x^3 + 2x^2 - 21x$ with a domain of all real numbers.

a. Use the first and second derivatives to find all local maximum and local minimum values.

First find the critical points to find the potential x values for the local max and min. The critical points are where the first derivative is equal to zero.

$$f'(x) = x^2 + 4x - 21 = 0$$
$$(x + 7)(x - 3) = 0$$
$$x = -7 \text{ or } x = 3$$

We find the second derivative to determine if the graph has a local max or min at these values:

Practice Final Exam Questions

$$f''(x) = 2x + 4$$
$$f''(-7) = -10$$
$$f''(3) = 10$$

So the graph is concave down at x=-7 so that is a local max.(f(-7)=130.67.) and concave up at x=3 so that is a local min (f(3)=-36)

$$(-7, 130.67) \textit{is a local max}$$
$$(3, -36) \textit{is a local min}$$

b. Where does $f(x)$ have an inflection point?

An inflection point occurs when the concavity changes – this happens when the second derivative equals zero:

$$2x + 4 = 0$$
$$x = -2$$

We already know that the concavity changes on either side of -2 so this must be an inflection point .f(-2)=47.333

$$f(x) \textit{has an inflection point at } (-2, 47.333)$$

c. On what intervals is $f(x)$ increasing?

d. On what intervals is $f(x)$ decreasing?

f(x) is increasing where the first derivative is positive and decreasing where it is negative. We know that the first derivative is zero at -7 and 3 so we need to examine the intervals x<-7, -7<x<3 and x>3

For x<-7 we can check any value: f'(-10) = 39 so the derivative is POSITIVE for x < -7 which means f(x) is INCREASING for x < -7

For -7 < x < 3 let's just check 0: f'(0) = -21 so the derivative is NEGATIVE for -7 < x < 3 which means f(x) is DECREASING for -7 < x < 3

For x>3 let's check f'(10)=119 so the derivative is POSITIVE for x >3 which means f(x) is INCREASING for x >3

e. On what intervals is $f(x)$ concave up?

f. On what intervals is $f(x)$ concave down?

The graph is concave up when the derivative is increasing and concave down when the derivative is decreasing.

The derivative is increasing when the second derivative is positive and decreasing when the second derivative is negative.

$$f''(x) = 2x + 4$$

We know that f''(x)=0 at -2 so on one side of -2 it is positive and on the other side it is negative so we just need to check. Let's just check x=0: f''(0)=4 so that means the second derivative is positive for x>-2 which means

f(x) is concave up for x>-2
f(x) is concave down for x<-2

g. Using the above information sketch a graph of $f(x)$.