Let \( X \) be a discrete random variable. 

- p.m.f. \( f(x) = P(X = x) \)
- \( F(x) = P(X \leq x) = \sum_{k \in S, k \leq x} f(x) \)
- \( E[u(x)] = \sum_{x \in S} u(x)f(x) \)
- \( \mu = E(X) = \sum_{x \in S} xf(x) \)
- \( \sigma^2 = E((x - \mu)^2) = E(X^2) - \mu^2 \)
- \( M(t) = E(e^{tx}) \)
- \( M'(0) = \mu \)
- \( M''(0) - \mu^2 = \sigma^2 \)

Let \( X \) be a continuous random variable. 

- p.d.f. \( f(x) \)
- \( F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx \)
- \( E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx \)
- \( \mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \)
- \( \sigma^2 = E((x - \mu)^2) = E(X^2) - \mu^2 \)
- \( M(t) = E(e^{tx}) \)
- \( M'(0) = \mu \)
- \( M''(0) - \mu^2 = \sigma^2 \)

\( \pi_p \) is the \((100p)th\) percentile if \( F(\pi_p) = \int_{-\infty}^{\pi_p} f(x)dx = p \)

Suppose you roll a fair 10-sided die (with numbers 1-10) over and over. Define a “success” or “change” as rolling a 6. Suppose on average that you roll twelve 6's every minute. For each distribution write questions that would be solved using a random variable with that distribution. Write the corresponding p.d.f or p.m.f. For each write one question solved by \( P(X \leq x) \) or \( P(X = x) \) and one solved by \( P(X \geq x) \). Hint: you might find tables XII and XIII on pages 596–597 of your book helpful.

### The Binomial Distribution

\( P(X \leq x) \) or \( P(X = x) \): What is the probability that I will roll five 6's in 20 rolls?

\( X \sim b(20, 0.1) \).

\[ P(X = 5) = \binom{20}{5} (0.1)^5 (0.9)^{15} = 0.0319 \]

OR use the binomial table: \( P(X = 5) = P(X \leq 5) - P(X \leq 4) = 0.9887 - 0.9568 = 0.0319 \)

\( P(X \geq x) \): What is the probability that I will roll at least five 6's in 20 rolls?

\[ P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.9568 = 0.0432 \) (using binomial table; \( n = 20, x = 4 \))
**The Geometric Distribution**

$P(X \leq x)$ or $P(X = x)$: What is the probability that I will need to roll at most four times before I get my first six?

$f(x) = (0.9)^{x-1}(0.1)$

$P(X \leq 4) = f(0) + f(1) + f(2) + f(3) + f(4) = (0.1) + (0.9)(0.1) + (0.9)^2(0.1) + (0.9)^3(0.1) = 0.3439$

OR I could use the formula on page 93 ($P(X \leq k) = 1 - q^k$): $P(X \leq 4) = 1 - (0.9)^4 = 0.3439$

$P(X \geq x)$: What is the probability that I will need to roll 3 or more times before I see my first six?

$P(X \geq 3) = f(3) + f(4) + \ldots$ This would be an infinite sum so I would do one of the following:

$P(X \geq 3) = 1 - P(X \leq 2) = 1 - f(0) - f(1) - f(2)$

or use the formula on page 93 ($P(X > k) = q^k$): $P(X \geq 3) = P(X > 2) = (0.9)^2 = 0.81$

**The Negative Binomial Distribution**

$P(X \leq x)$ or $P(X = x)$: What is the probability that I see my third six on my seventh roll?

$f(x) = \binom{x - 1}{2}(0.1)^3(0.9)^{x-3}; \ x = 3, 4, 5, \ldots$

$P(x = 7) = f(7) = \binom{6}{2}(0.1)^3(0.9)^4 = 0.0098$

$P(X \geq x)$: What is the probability that I need more than 8 rolls before I see my third six?

$P(X > 8) = 1 - P(X \leq 8) = 1 - (f(0) + f(1) + f(2) \cdots f(8))$
The Poisson Distribution

\[ P(X \leq x) \text{ or } P(X = x): \text{ What is the probability that I will roll 80 sixes in 5 minutes?} \]

\[ X \text{ is Poisson with } \lambda = 5 \times 12 = 60; \quad f(x) = \frac{(60)^x e^{-60}}{x!} \]

\[ f(80) = \frac{60^{80} e^{-60}}{80!} = \text{too big for my calculator but you get the idea} \]

\[ P(X \geq x): \text{ What is the probability that I will get more than 5 sixes in 15 seconds?} \]

Now \( \lambda = 12/4 = 3 \)

\[ P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.815 = 0.185 \text{ using the Poisson table with } \lambda = 3; \ x = 4 \]

The Exponential Distribution

\[ P(X \leq x) \text{ or } P(X = x): \text{ What is the probability that I will get my first six before 10 seconds?} \]

\[ X \text{ is exponential with } \theta = 5 \text{ (Note our units are seconds)} \quad f(x) = \frac{1}{\theta} e^{-x/\theta} \]

\[ P(X \leq 10) = \int_0^{10} \frac{1}{5} e^{-x/5} dx \]

or use the formula on page 143 \[ P(X \leq 10) = F(10) = 1 - e^{-10/5} = 0.865 \]

\[ P(X \geq x): \text{ What is the probability that I will get my first six after 5 seconds?} \]

\[ P(X \geq 5) = \int_5^{\infty} \frac{1}{5} e^{-x/5} dx \]

or use the formula on page 143 \[ P(X \geq 5) = 1 - P(X \leq 5) = 1 - (1 - e^{-5/5}) = 0.368 \]
The Gamma Distribution

$P(X \leq x)$ or $P(X = x)$: What is the probability that I will have to wait no more than 20 seconds to see two 6’s?

$X$ is gamma with $\theta = 5$ and $\alpha = 2$

$P(X \leq 20) = F(20) = 1 - \sum_{k=0}^{1} \frac{(20)^k}{k!} e^{-20/5} = 1 - e^{-4}(1 + 4) = 1 - 0.092 = 0.908$

Note that $\sum_{k=0}^{1} \frac{(20)^k}{k!} e^{-20/5}$ is a sum of a Poisson variable with $\lambda = \frac{20}{5} = 4$ and the sum is $P(x \leq 1)$ so I could just use the Poisson table to find the value of the sum by looking up $x = 1$ in the table under $\lambda = 4$

$P(X \geq x)$: What is the probability that I have to wait more than 30 seconds to see five 6’s?

$P(X > 30) = 1 - F(30) = 1 - (1 - \sum_{k=0}^{4} \frac{(30)^k}{k!} e^{-30/5}) = \sum_{k=0}^{4} \frac{(30)^k}{k!} e^{-30/5} = 0.285$

Note: I can solve this sum by plugging in the values and adding them up or by noticing that it is a sum of a Poisson variable with $\lambda = \frac{30}{5} = 6$ and looking up $x = 4$ in the Poisson table. Note that the value of $x$ to look up in the Poisson table is the number at the top of the summation.

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Poisson approximation to binomial...

Suppose I roll the die 120 times. Find the probability that I will get at most 15 6’s.

Since $n$ is large and $p$ is small I can use the Poisson approximation to binomial: Use $\lambda = \mu = np = 120 \times (0.1) = 12$.

Using the Poisson table $P(X \leq 15) = 0.844$