Math 344 Chapter 2 - Is it a group?

**Directions:** For each item, determine if the given set and operation form a group. If it is a group, prove it and determine if the group is abelian. If it is not a group, determine which properties fail and give counterexamples to each property that fails. If it is not a group, can you modify the set (e.g. throw out or add some elements) to make the set and operation a group, explain?

- **Definition of Group** Let $G$ be a set and $\circ$ a binary operation on $G$ that assigns to each ordered pair $(a, b)$ of elements of $G$ an element in $G$ denoted by $a \circ b$. We say that $G$ is a **GROUP** under the operation $\circ$ if the following three properties are satisfied:
  1. **Associative**. The operation is associative on the set $G$: $\forall a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$
  2. **Identity**. There is an element $e \in G$ (called the identity element) such that $a \circ e = e \circ a = a$ for all $a \in G$.
  3. **Inverses**. For each $a \in G$ there is an element $b \in G$ such that $a \circ b = b \circ a = e$ ($b$ is called the inverse of $a$).
- Note that "hidden" in this definition is that the $\circ$ is CLOSED on $G$. Don't forget to check that property.
- We say the group is **Abelian** if it is commutative: $a \circ b = b \circ a \ \forall a, b \in G$.

1. Let the set be $Z_7^* = \{1, 2, 3, 4, 5, 6\}$ and the operation $\times_7$ (multiplication modulo 7). (Hint: Start by making a Cayley table.)

2. Let the set be $Z_9^* = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the operation $\times_9$. (Hint: Start by making a Cayley table.)

3. Let the set be all $2 \times 2$ matrices: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ with the operation of matrix multiplication.

4. For a fixed pair $a$ and $b$, define $T_{a,b} : \mathbb{R} \to \mathbb{R}$ by $T_{a,b}(x) = ax + b$. Let the set be $G = \{T_{a,b} \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$ and let the operation be function composition.