Math 344 Chapter 2 - Elementary Properties of Groups

- **Definition of Group** Let G be a set and $\circ$ a binary operation on G that assigns to each ordered pair (a, b) of elements of G an element in G denoted by $a \circ b$. We say that G is a **GROUP** under the operation $\circ$ if the following three properties are satisfied:

  1. **Associative**. The operation is associative on the set G: $\forall a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$
  2. **Identity**. There is an element $e \in G$ (called the identity element) such that $a \circ e = e \circ a = a$ for all $a \in G$.
  3. **Inverses**. For each $a \in G$ there is an element $b \in G$ such that $a \circ b = b \circ a = e$ (b is called the inverse of a).

- Note that "hidden" in this definition is that the $\circ$ is CLOSED on G. Don't forget to check that property.

- We say the group is **Abelian** if it is commutative: $a \circ b = b \circ a \ \forall a, b \in G$.

Directions: Prove each of the following.

1. **Theorem 2.1 Uniqueness of the Identity** In a group G, there is only one identity element.

2. **Theorem 2.2 Cancellation** In a group G, the right and left cancellation laws hold; that is, $ba = ca$ implies $b = c$ and $ab = ac$ implies $b = c$.

3. **Theorem 2.3 Uniqueness of Inverses** For each element a in a group G, there is a unique element b in G such that $ab = ba = e$. (Unless otherwise noted, we will always assume e is the identity element of the group.)

4. **Theorem 2.4 Socks-Shoes Property** For group elements a and b, $(ab)^{-1} = b^{-1}a^{-1}$.