Math 344 Chapters 8 and 11 - External Direct Products and The Fundamental Theorem of Finite Abelian Groups

The **external direct product** of groups G_1, G_2,...,G_n, is the set of $n-tuples$ for which the i-th component is an element of G_i:

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, g_2, ..., g_n) \mid g_i \in G_i\}$$

The external direct product $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ is a group where:

- The operation is component-wise: $(g_1, g_2, ..., g_n)(h_1, h_2, ..., h_n) = (g_1h_1, g_2h_2, ..., g_nh_n)$ where g_ih_i is defined by the operation in G_i.
- The identity element is $e = (e_1, e_2, ..., e_n)$ where e_i is the identity element of G_i.
- $(g_1, g_2, ..., g_n)^{-1} = (g_1^{-1}, g_2^{-1}, ..., g_n^{-1})$ where g_i^{-1} is the inverse of g_i in G_i.

Theorem 8.1 The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the elements. In symbols,

$$|(g_1, g_2, ..., g_n)| = lcm(|g_1|, |g_2|, ..., |g_n|).$$

Theorem 8.2 Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

Corollary 1 An external direct product $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ of a finite number of finite cyclic groups is cyclic if and only if $|G_i|$ and $|G_j|$ are relatively prime for all $i \neq j$.

Corollary 2 Let $m = n_1n_2...n_k$. Then $\mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ if and only if n_i and n_j are relative prime for all $i \neq j$.

Theorem 11.1 (The Fundamental Theorem of Finite Abelian Groups) Every finite Abelian group is (isomorphic to) a direct product of cyclic groups of prime-power order. Moreover, the number of terms in the direct product and the orders of the cyclic groups are uniquely determined by the group.

Fact The direct product operation is commutative: $G \oplus H \cong H \oplus G$ ($\cong$ means isomorphic here).

Fact It is OK to "cancel": $G \oplus H \cong G \oplus K$ implies that $H \cong K$.

Fact: It can be shown that two finite Abelian groups are isomorphic if and only if they have the same number of elements of every order.

Recall:

- **Theorem 4.4** If d is a positive divisor of n, the number of elements of order d in a cyclic group of order n is $\phi(d)$.
- **Corollary to Theorem 4.4** In a finite group, the number of elements of order d is a multiple of $\phi(d)$.
- For any group,G, if $a \in G$ and $|a| = k$, then $\langle a \rangle$ is a cyclic subgroup of order k.
- Denote by $\cong$ "isomorphic"
- If we say "direct prodcut" we will mean "external direct product"

Last Homework of 344 - DUE Friday 12/1 by 4:30pm

1. What is the order of $\mathbb{Z}_3 \oplus \mathbb{Z}_9$?
2. What is the order of the element $(3, 3)$ in $\mathbb{Z}_3 \oplus \mathbb{Z}_9$?
3. How many elements order 10 are in $\mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$? How many cyclic subgroups of order 10?
4. How many elements of order 4 are in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$? How many cyclic subgroups of order 4?
5. Write down three elements of order 10 in $\mathbb{Z}_{10} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$.
6. Write down all finite abelian groups (up to isomorphism) of order 100.
7. Is $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2$? Justify.
8. Is $\mathbb{Z}_6 \oplus \mathbb{Z}_{35} \cong \mathbb{Z}_{14} \oplus \mathbb{Z}_{15}$? Justify.
9. Consider the group $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express G as a direct product of cyclic groups.
10. How many finite abelian groups (up to isomorphism) of order 72 have three elements of order 2? Justify.