RECONSTRUCTION OF 2-D SIGNALS FROM UNIONS OF SHIFTED LATTICES

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ABSTRACT

In this paper, we consider sampling theory for nonuniform sampling sets of minimal density which are structured. An efficient method for the reconstruction of band-limited discrete signals from sampling sets which are union of shifted lattices is developed. These sets are not necessarily periodic. A signal can be reconstructed from its samples provided the sampling set and the spectrum of the signal satisfy certain compatibility conditions. While explicit reconstruction formulas for unions of sampling lattices are possible, it is more convenient to use a recursive algorithm. A numerical example implemented in MATLAB is given.

1. INTRODUCTION

The reconstruction or approximation of a discrete signal from a set of finite samples is one of the major questions in signal analysis. If the sampling points are equally spaced, then the reconstruction of the signal can be accomplished by one of the many forms of the classical sampling theorem [7, 8, 6]. Sampling sets that are not equally spaced are called irregular or nonuniform sampling sets. We categorize these sampling sets into two classes, structured irregular sampling sets and nonstructured irregular sampling sets. Nonstructured irregular sampling sets in one and two dimensions have been considered extensively, see the reviews [2, 5, 9] as well as the references given there. An example of structured irregular sampling sets is what is known as periodic sampling. A periodic sampling sets is constructed of a finite number of nonequidistant sampling points that are shifted by the same period, see the reviews

[3, 4] as well as the references given there. A sampling set may also be a union of shifted lattices of different periods. Such a structured irregular sampling set is nonperiodic [10]. This case was also considered in the general setting of locally compact abelian group in [1]. In this paper, we present the results for specific case of 2-D discrete signals with nonperiodic structured irregular sampling sets. In the last section, we give a numerical example implementing our recursive algorithm in MATLAB for the group $\mathbb{Z}_{512} \times \mathbb{Z}_{512}$.

Let $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the integers, reals, and complex numbers, respectively. We use $\jmath$ for the imaginary unit. A signal is denoted by a small letter, its Fourier transform is denoted by the corresponding capitalized letter (e.g., X is the Fourier transform of the signal x.) A discrete 2-D signal of size $L \times L$ is a rectangular matrix x of size $L \times L$ with real or complex entries $x(k, l) \in \mathbb{C}$ for $k, l = 0, 1, \ldots, L - 1$. For numerical implementation, we identify the index set $\{0, \ldots, L-1\} \times \{0, \ldots, L-1\}$ with the finite cyclic group $G = \mathbb{Z}_L \times \mathbb{Z}_L$ where $\mathbb{Z}_L$ means $\mathbb{Z}$ with addition modulo L. Hence the signal x is understood as two-dimensional bi-periodic function with period L. The character group $\widehat{G}$ can be defined by $\widehat{G} = \{\nu/L, \ \nu = 0, \ldots, L-1\} \times \{\mu/L, \ \mu = 0, \ldots, L-1\}$ with addition modulo one. The discrete two-dimensional Fourier transform on G is defined by

$$X(m, n) = \sum_{k=0}^{L-1} \sum_{l=0}^{L-1} x(k, l) e^{-2\pi \jmath (mk + nl)/L}$$

for $m, n = 0, \ldots, L - 1$. The l^2-norm given by $||x|| = \left(\sum_{k=0}^{L-1} \sum_{l=0}^{L-1} |x(k, l)|^2 \right)^{1/2}$ represents the energy of x.

A sampling lattice is a set of points defined by $H = W\mathbb{Z}^2 \cap (\mathbb{Z}_L \times \mathbb{Z}_L)$ where W is a 2 by 2 nonsingular matrix of the form

$$W = \left[\begin{array}{cc} h_1 & 0 \\ 0 & h_2 \end{array} \right]$$

with $h_i \in \mathbb{Z}_L$ such that h_i divides L for $i = 1, 2$. That is $H = \{kh_1 : k = 0, \ldots, r_1 = L/h_1 - 1\} \times \{lh_2 : l = 0, \ldots, r_2 = L/h_2 - 1\}$. We use the notation $H = \langle h_1, h_2 \rangle$ indicating that H is generated

by h_1 and h_2 in the horizontal and the vertical directions respectively. The lattice $H^\perp = W^{-T}\mathbb{Z}^2$ is called the dual lattice with respect to H. Hence $H^\perp = \{\nu/h_1,\ \nu = 0,\dots,h_1-1\} \times \{\mu/h_2,\ \mu = 0,\dots,h_2-1\}$. We define a fundamental domain of $H^\perp$ to be given by $R = \{\nu/L,\ \nu = 0,\dots,L/h_1 - 1\} \times \{\mu/L,\ \mu = 0,\dots,L/h_2-1\}$, which can be viewed as essential support of X. The sampling set H has minimal density in the sense that there are as many sampling points as there are points in the set R.

Let $K \subset \widehat{G}$ to be a finite union of non-overlapping rectangles. We define the space of discrete band-limited signals with finite energy by

$$\mathcal{B}_K = \{x \in G : X(m,n) = 0 \text{ for } (m/L,n/L) \notin K\}.$$

Note that in our definition, the support of the Fourier transform of a signal x maybe comprised of finite union of contiguous rectangles. Clearly, this definition does not exclude hypercubes. Also, note that the standard definition of band-limited images with bandwidth $M \times M$, where $0 < M < L/2$ is included in our definition.

The discrete version of the classical sampling theorem in 2-D is as follows.

Theorem 1.1 Let H be a sampling lattice and R a fundamental domain of $H^\perp$. Let $x \in L_2(G)$ be a signal such that $x \in \mathcal{B}_R$. Then

$$x(s,t) = \frac{1}{L^2} \sum_{k=0}^{r_1} \sum_{l=0}^{r_2} x(h_1 k, h_2 l)\, \varphi_R(s - h_1 k, t - h_2 l)$$

$$(1)$$

where $r_i = L/h_i - 1$, $i = 1,2$ and

$$\varphi_R(x,y) = h_1 h_2 \sum_{m=0}^{r_1} \sum_{n=0}^{r_2} e^{2\pi J(mx+ny)/L}.$$

Note that if L/h_i, $i = 1,2$ is odd, then using the periodicity of x and X, negative indices can be admitted and hence the fundamental domain R can be shifted so that it is symmetrical around the origin. In that case, the function φ is the product of two sinc functions.

Suppose H is a sampling lattice and R a fundamental domain of $H^\perp$. If $x \in L_2(G)$ is a signal such that $x \in \mathcal{B}_K$ with R a proper subset of K, then x cannot be reconstructed from the sample values of the set H. However, if K can be covered by a finite union of shifts of R, using the reconstruction formula (1) we obtain a signal $S_M x$ that coincides with the signal x over the sampling lattice H. The corollary explains this case. It is used in the reconstruction theorems given in the next section.

Corollary 1.2 Let H be a lattice and R a fundamental domain of $H^\perp$. Let $x \in L_2(G)$ be a signal such that $x \in \mathcal{B}_K$. Assume that there is $P < \infty$ such that $K \subseteq \cup_{i=1}^P (\eta_i + R)$ with $\eta_1,\dots,\eta_P$ distinct elements of $H^\perp$. For $s_0, t_0 \in \mathbb{Z}_L$, let $M = (s_0, t_0) + \langle h_1, h_2 \rangle$ be a shift of H. Then the function $S_M x$ defined by

$$S_M x(s,t) = \frac{1}{L^2} \sum_{k=0}^{r_1} \sum_{l=0}^{r_2} x(s_0 + h_1 k, t_0 + h_2 l) \times$$
$$\varphi_R(s - s_0 - h_1 k, t - t_0 - h_2 l) \quad (2)$$

is square integrable on G and satisfies $S_M x(y,z) = x(y,z)$ for all $(y,z) \in M$.

The next lemma plays a fundamental role in our sampling theorems. We consider the case where support of the Fourier transform is not contained in a fundamental domain of $H^\perp$, but is contained in the union of a fundamental domain and a finite union of its translates.

Lemma 1.3 Let H be a lattice and R a fundamental domain of $H^\perp$, with $(0,0) \neq \eta' = (\nu'/h_1, \mu'/h_2) \in H^\perp$. Let $K = R \cup (\eta' + K')$ where $K' \subset R$. Assume that $x \in L_2(G)$ vanishes on the coset $(s_0, t_0) + \langle h_1, h_2 \rangle$ and that $x \in \mathcal{B}_K$. Then

$$x(s,t) = w(s,t)\left(1 - e^{2\pi J\left(\frac{(s-s_0)\nu'}{h_1} + \frac{(t-t_0)\mu'}{h_2}\right)}\right) \quad (3)$$

where $w \in L_2(G)$ and $\widehat{w} \in \mathcal{B}_{K'}$.

2. SAMPLING THEOREMS

The reconstruction of the signal x can now be reduced to the reconstruction of the signal w using Lemma 1.3 as illustrated in the following theorem.

Theorem 2.1 Let H be a lattice and R a fundamental domain of $H^\perp$, with $(0,0) \neq \eta' = (\nu'/h_1, \mu'/h_2) \in H^\perp$. Let $K = R \cup (\eta' + K')$ where $K' \subset R$. Assume that $x \in L_2(G)$ such that $x \in \mathcal{B}_K$. Let $M' \subset G$ be a sampling lattice such that signals $w \in L_2(G)$ with $\widehat{w} \in \mathcal{B}_{K'}$ can be constructed form their samples $w(y',z')$, $(y',z') \in M'$. Let (s_0, t_0) be such that

$$\frac{(y'-s_0)\nu'}{h_1} + \frac{(z'-t_0)\mu'}{h_2} \neq 0 \text{ for all } (y',z') \in M' \quad (4)$$

Then x can be reconstructed form its samples $x(y,z)$, $(y,z) \in M \cup M'$, where $M = (s_0, t_0) + \langle h_1, h_2 \rangle$ by the formula

$$x(s,t) = S_M x(s,t) +$$
$$w(s,t)\left(1 - e^{2\pi J\left(\frac{(s-s_0)\nu'}{h_1} + \frac{(t-t_0)\mu'}{h_2}\right)}\right) \quad (5)$$

Note that the theorem provides a general method to generate new sampling theorems from known ones. Given a sampling theorem for a set K', we can obtain one for $K = R \cup (\eta' + K')$ by adding a shift of the sampling set H to the original sampling lattice M'. Provided the condition (4) is satisfied, the primary limitation is the requirement that H must be sufficiently dense so that $K' \subset R$.

To obtain our general sampling theorem, we apply Theorem 2.1 repeatedly. This gives a recursive algorithm to construct x from its samples on shifts of sampling lattices $H_1, H_2, \ldots, H_N$, provided that the sampling lattice $H_j = \langle h_{1,j}, h_{2,j} \rangle$ and the set K satisfy certain compatibility conditions. These conditions are given in the following definition which presents the structure of the set K we consider as support of the Fourier transform $\widehat{x}$. This structure is the generalization of the structure of the set K in Lemma 1.3.

Definition 2.2 Let $H_1, \ldots, H_N$ be lattices with corresponding fundamental domains R_i of $H_i^{\perp}$. We call $K \subset \widehat{G}$ an admissible subset of $\widehat{G}$ with respect to $H_1, \ldots, H_N$ if there are subsets $K_1, \ldots, K_N$ of $\widehat{G}$ such that the following conditions hold:
   i) $K_1 = R_1$,
   ii) $K_j \subset R_{j+1}$, $j = 1, \ldots, N-1$,
   iii) $K_{j+1} = R_{j+1} \cup (\eta_{j+1} + K_j)$ with $0 \neq \eta_{j+1} = (\nu/h_{1,j+1}, \mu/h_{2,j+1}) \in H_{j+1}^{\perp}$, $j = 1, \ldots, N-1$
   iv) $K_N = K$.

Observe that because of conditions ii) and iii) each intermediate set K_{j+1} has the structure of the set K in Lemma 1.3 with $R = R_{j+1}$, $K' = K_j$ and $\eta' = \eta_{j+1}$. The above conditions imply in particular that $R_1 \subset R_2 \subset \ldots \subset R_N$, so that the subgroups H_j are ordered by increasing density.

The following theorem is the main result.

Theorem 2.3 Suppose that K is an admissible subset of G with respect to the sampling lattices $H_1, \ldots, H_N$, with R_j, K_j, η_j, $j = 1, \ldots, N$ as in Definition 2.2. Let $M_j = (s_j, t_j) + \langle h_{1,j}, h_{2,j} \rangle$, $j = 1, \ldots, N$ be such that for $N > 1$

$$\frac{(y - s_j)\nu}{h_{1,j}} + \frac{(z - t_j)\mu}{h_{2,j}} \neq 0 \ \text{ for } \ (y, z) \in \bigcup_{k=1}^{j-1} M_k \quad (6)$$

with $j = 2, \ldots, N$. Let $x \in L_2(G)$ with $x \in \mathcal{B}_K$. Then there are continuous functions $x_j \in L_2(G)$ such that $x_j \in \mathcal{B}_{K_j}$, and for all $(s, t) \in G$:

$$x_1(s, t) = S_{M_1} x_1(s, t),$$
$$x_j(s, t) - S_{M_j} x_j(s, t) = x_{j-1}(s, t) \times$$
$$\left(1 - e^{2\pi J \left(\frac{(s-s_j)\nu}{h_{1,j}} + \frac{(t-t_j)\mu}{h_{2,j}} \right)} \right), \quad j = 2, \ldots, N,$$
$$x_N(s, t) = x(s, t).$$

Using this recursion, the function f can be reconstructed from sampled values $x(y, z)$, $(y, z) \in \bigcup_{k=1}^{N} M_k$.

The theorem establishes the following recursive algorithm for reconstruction of x from sampled values $x(y, z)$, $(y, z) \in \bigcup_{k=1}^{N} M_k$:

Algorithm 2.4 :

IF $N = 1$ THEN $x(s, t) = S_{M_1} x(s, t)$.

ELSE

   Compute

$$g(y, z) = \frac{x(y, z) - S_{M_N} x(y, z)}{\left(1 - e^{2\pi J \left(\frac{(y-s_N)\nu}{h_{1,N}} + \frac{(z-t_N)\mu}{h_{2,N}} \right)} \right)},$$

   $(y, z) \in \bigcup_{k=1}^{N-1} M_k$.

   Invoke the algorithm to compute $g(s, t)$, $(s, t) \in G$ from the computed values $g(y, z)$, $(y, z) \in \bigcup_{k=1}^{N-1} M_k$.

$$x(s, t) = g(s, t) \left(1 - e^{2\pi J \left(\frac{(s-s_N)\nu}{h_{1,N}} + \frac{(t-t_N)\mu}{h_{2,N}} \right)} \right)$$
$$+ S_{M_N} x(s, t), \quad (s, t) \in G.$$

END

Clearly, Theorem 2.3 also gives rise to explicit formulas(see, e.g.,[1] for the case $N = 2$.) However, as N increases these formulas seem to become too complicated to be useful. On the other hand, Algorithm 2.4 is very easy to program if the programming language allows for recursive function calls; see, e.g., the MATLAB M-file bfmethod.m in the proceedings CD-ROM.

3. NUMERICAL EXAMPLE

In this section we illustrate Theorem 2.3 and Algorithm 2.4 with an example implemented in MATLAB. The parameters are specified and explained in the driver routine npr2d.m. This routine generates the function to be reconstructed by randomly specifying its non-zero Fourier coefficients, cf. [5]. The recursive algorithm is implemented in the function M-file bfmethod.m. The function M-file SM.m computes $S_M x$. The function M-file spect.m computes the domain of the Fourier transform of the function based on Definition 2.2. Note that depending on the values of $\eta's$, the set K may be a hypercube or union of several contiguous sets. In order to keep the code readable the simplifying assumption was made that all fundamental domains R_j are of the form given above.

In the code given in the proceedings CD-ROM, the parameters to be specified by the user are set as follows: We specified $L = 512$. As the set K we consider the union of the two contiguous sets $\{0, \dots, 127\}/L \times \{0, \dots, 127\}/L$, $\{256, \dots, 383\}/L \times \{128, \dots, 191\}/L$, and $\{256, \dots, 319\}/L \times \{192, \dots, 255\}/L$, giving a total of 28672 points. Now choose $H_1 = \langle 8, 8 \rangle$, $H_2 = \langle 4, 8 \rangle$, and $H_3 = \langle 4, 4 \rangle$. Note that this sampling set has minimal density in the sense that there are as many sampling points as there are points in the set K, i.e., 28672. Hence, about 89% of the data is discarded. We chose $R_1 = (1/L) \cdot (\{0, \dots, 63\} \times \{0, \dots, 63\})$, $R_2 = (1/L) \cdot (\{0, \dots, 127\} \times \{0, \dots, 63\})$, and $R_3 = (1/L) \cdot (\{0, \dots, 127\} \times \{0, \dots, 127\})$. Choosing $\eta_2 = (0/L, 64/L)$ and $\eta_3 = (256/L, 128/L)$ we have $K = R_3 \cup (\eta_3 + R_2) \cup ((\eta_3 + \eta_2) + R_1)$, and the sets $K_1 = R_1$, $K_2 = R_2 \cup (\eta_2 + R_1)$, and $K_3 = K = R_3 \cup (\eta_3 + K_2)$ satisfy the conditions of Definition 2.2. The shifts (s_j, t_j) have to be chosen such that the sampling conditions (6) are satisfied. According to [1], this condition is equivalent to the cosets M_1, M_2, M_3 being mutually disjoint. Two cosets $(s_i, t_i) + \langle h_{1,i}, h_{2,i} \rangle$ and $(s_j, t_j) + \langle h_{1,j}, h_{2,j} \rangle$ will intersect if and only if the difference $s_i - s_j$ is an integer multiple of the greatest common divisor of $h_{1,i}$ and $h_{1,j}$ and the difference $t_i - t_j$ is an integer multiple of the greatest common divisor of $h_{2,i}$ and $h_{2,j}$. Hence the conditions (6) require in this particular example that $s_1 - s_2$, $s_2 - s_3$, $s_1 - s_3$, $t_1 - t_3$, and $t_2 - t_3$ should not be a multiple of 4 and $t_1 - t_2$ should not be a multiple of 8. The relative errors in our numerical tests varied with the random signal, but stayed below $3.e - 13$. In order to assess the stability of the algorithm we computed as a comparison the relative error resulting from taking the FFT of the signal f and then reconstructing by an inverse FFT. The relative error resulting from this very stable procedure was about $2.e - 14$, indicating that our algorithm is stable in this case.

If δ_1 and δ_2 denote the maximum gap in the sampling set in the $x-$ and $y-$direction respectively, then in our example $\delta_1 = \delta_2 = 8$. This can be compared to the optimal regular sampling distance, i.e., the spacing of the smallest sampling lattice H such that the set K is a subset of a fundamental domain of $H^{\perp}$ and Theorem 1.1 can be applied. For this example the smallest feasible subgroup is $H = \langle 2, 2 \rangle$ that has 65536 elements with the spacing of $\delta_1 = \delta_2 = 2$.

Note that our sampling theory is independent of the shape of the Fourier transform of the function as being a hypercube or not. In the example above, if we choose $H_1 = H_2 = \langle 8, 4 \rangle$, $\eta_2 = (64/L, 0/L)$ and $\eta_3 = (128/L, 0/L)$, we have the spectrum K as a rectangular domain of length 256 and width 128.

4. SUMMARY

In this paper, we have presented a sampling theorem for nonuniform structured sampling lattices. Even though, one can find explicit formulas for reconstruction, Theorem 2.3 produces a recursive algorithm that is most convenient for any programming language with recursive function calls. We have recently produced results that relaxes the restrictive condition of Lemma 1.3 substantially. This will allow for less restrictive reconstruction methods. These results will be published in near future.

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