DECOMPOSITION OF BANDLIMITED FUNCTIONS ON LATTICES

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Abstract. We consider a bandlimited function that vanishes on a sampling lattice with the assumption that the disjointedness property (of the support K of the Fourier transform of the function with respect to the elements of the annihilator subgroup of the sampling lattice) does not hold. In this case, the function does not vanish, but it decomposes into finitely many continuous functions with their Fourier transforms supported on the equivalence classes of the set K.

1. Introduction

The classical sampling theorem permits reconstruction of a bandlimited function from its values on a set of equidistant points on the real line $\mathbb{R}$ [7, 8, 10]. Kluvánek's important generalization results from replacing $\mathbb{R}$ by an arbitrary locally compact abelian (LCA) group G [6]. The sampling set is then a coset of a closed subgroup (lattice) H of G. Using the classical sampling theorem, the reconstruction of the function f is possible if the support K of the Fourier transform of f satisfies the condition that the sets $K + \eta$ be mutually disjoint for all $\eta \in H^{\perp}$, where $H^{\perp}$ represents the annihilator subgroup of H. Specifically, if a function f vanishes on a lattice H and the support of its Fourier transform K satisfies the disjointedness property, then f must vanish almost everywhere. The present work investigates the case where the function f vanishes on H but the translated sets $K + \eta$ are not disjoint. Seminal results for the case $G = \mathbb{R}$ have been derived by Walnut [9] in Lemma 3.1.

The paper is organized as follows. In section 2 we begin with a review of basic definitions and facts about Fourier analysis on locally compact abelian groups. This general setting facilitates obtaining results which cover a large class of applications. We will only use the basic concepts of the theory. The main results are developed and illustrated with examples in section 3. There, we consider the following problem. If H is a closed subgroup (lattice) of a LCA group G and K is a compact subset of the character group of G, then K can be decomposed into equivalence classes K_j; see [3, pp.314-315]. Let f be a continuous function defined on the LCA group G with its Fourier transform supported on K. Assume that f vanishes on a subgroup H. Then f can be decomposed into continuous functions f_j with $\hat{f}_j$ vanishing almost everywhere outside K_j.

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2. Standard definitions and facts

Let $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the integers, reals, and complex numbers, respectively. Let G denote a locally compact abelian (LCA) group written additively. The character group $\widehat{G}$ consists of the continuous homomorphisms of G into the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The value of the character $\xi \in \widehat{G}$ at the point $x \in G$ is written $\langle x, \xi \rangle$. $\widehat{G}$ has a natural addition and a natural topology relative to which it is also an LCA group. On every LCA group there exists a non-negative regular measure m_G, the so-called Haar measure of G, which is not identically zero and translation invariant. The Haar measure is uniquely determined up to multiplication by a constant. $L_p(G)$ denotes the space of all Borel functions on G such that $\| f \|_p = \left(\int_G |f(x)|^p \, dm_G(x) \right)^{1/p}$ is finite.

The Fourier transform of a function $f \in L_1(G)$ is the continuous function $\hat{f}$ on $\widehat{G}$ defined by

$$\hat{f}(\xi) = \int_G f(x) e^{-2\pi i \langle x, \xi \rangle} \, dm_G(x).$$

We will always normalize the Haar measure on $\widehat{G}$ such that the following holds.

Theorem 2.1. *(Fourier inversion formula) If $f \in L_1(G)$ is continuous and $\hat{f} \in L_1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, dm_{\widehat{G}}(\xi) = (\hat{f})^\wedge(-x). \tag{2.1}$$

The Fourier transform can be extended to a linear isomorphism of $L_2(G)$ onto $L_2(\widehat{G})$ by means of the Plancherel Theorem (cf. [5, Sec. 31.18]).

Let H be a closed subgroup of an LCA group G. The annihilator of H is the set $H^\perp \subset \widehat{G}$ given by $H^\perp = \{\eta \in \widehat{G} : \ \langle y, \eta \rangle = 0 \text{ for all } y \in H\}$. $H^\perp$ is a closed subgroup of $\widehat{G}$ and is isomorphically homeomorphic to the character group of G/H, i.e., $H^\perp = (G/H)^\wedge$. Furthermore we have that $(H^\perp)^\perp = H$, and $\widehat{H} = \widehat{G}/H^\perp$.

Definition 2.2. A closed discrete subgroup H of G such that $H^\perp$ is also discrete is called a *lattice*. A measurable subset R of $\widehat{G}$ such that every $\xi \in \widehat{G}$ can be uniquely written as $\xi = \rho + \eta$, where $\rho \in R$ and $\eta \in H^\perp$ is called a *fundamental domain* of $H^\perp$.

Convention 2.3. Throughout this paper we assume that m_G is given and normalize the Haar measure on $\widehat{G}$ such that the Fourier inversion formula (2.1) holds. For a lattice H and R a fundamental domain of $H^\perp$ we normalize the Haar measures on H, $H^\perp$, and $\widehat{G}/H^\perp$ such that

(i) m_H equals the counting measure,
(ii) $m_{H^\perp}$ equals $m_{\widehat{G}}(R)$ times the counting measure, and
(iii) $m_{\widehat{G}/H^\perp}(\widehat{G}/H^\perp) = 1$.

The following version of the Poisson summation formula is given by Gröchenig in [4, p. 217].

Theorem 2.4. *Suppose H is an admissible lattice with R as a fundamental domain of $H^\perp$. If $\hat{f} \in L_1(\widehat{G})$, then the periodization $R_{H^\perp}\hat{f}(\xi + H^\perp) = m_{\widehat{G}}(R) \sum_{\eta \in H^\perp} \hat{f}(\xi + \eta)$ is in $L_1(\widehat{G}/H^\perp)$ and for $y \in H$*

$$(R_{H^\perp}\hat{f})^\wedge(y) = f(y). \tag{2.2}$$

If furthermore $\sum_{y\in H}|f(y)|^2<\infty$*, then* $R_{H^\perp}\hat{f}\in L_2(\widehat{G}/H^\perp)$ *and*

$$R_{H^\perp}\hat{f}(\xi+H^\perp)=m_{\widehat{G}}(R)\sum_{\eta\in H^\perp}\hat{f}(\xi+\eta)=\sum_{y\in H}f(y)e^{-2\pi i\langle y,\xi\rangle} \tag{2.3}$$

where the equality (2.3) holds almost everywhere and the right hand side converges in $L_2(\widehat{G}/H^\perp)$*.*

Proof. For a proof, see [4, p. 217]. ◻

3. Decomposition Theorems

In the introduction, we mentioned a decomposition of K into equivalence classes K_j. This decomposition plays an important role throughout this work. To motivate the abstract theory, we will give an example and then present the decomposition.

Example 3.1. : Let $G=\mathbb{R}$, $H=\mathbb{Z}$. Then $\widehat{G}=\mathbb{R}$, $H^\perp=\mathbb{Z}$, and $\widehat{H}=\mathbb{R}/\mathbb{Z}=\mathbb{T}$. Hence fundamental domain R of $H^\perp$ can be chosen to be [0,1). Thus $m_{\widehat{G}}(R)=1$. Assume that $K=[0,3/2]$, and $f\in L_2(\mathbb{R})$ such that the hypothesis of Theorem 2.4 holds. Define $F(\xi)=\sum_{n\in\mathbb{Z}}f(n)e^{-2\pi in\xi}$. Following Poisson summation formula (2.3), we have

$$F(\xi)=\sum_{n\in\mathbb{Z}}f(n)e^{-2\pi in\xi}=\sum_{n\in\mathbb{Z}}\hat{f}(\xi+n) \tag{3.1}$$

where $\xi\in\widehat{G}$ and $\eta\in H^\perp$. Since $\hat{f}$ vanishes outside K, (3.1) becomes

$$F(\xi)=\begin{cases}\hat{f}(\xi)+\hat{f}(\xi+1), & \text{if}\xi\in[0,1/2]=K_2,\\ \hat{f}(\xi), & \text{if}\xi\in(1/2,1)=K_1,\\ \hat{f}(\xi-1)+\hat{f}(\xi), & \text{if}\xi\in[1,3/2]=K_3\end{cases}$$

For $\xi\in K_1=(1/2,1)$, $\hat{f}(\xi)$ can be recovered from $f(n)$ with $n\in\mathbb{Z}$. However, for $\xi\in K_2\cup K_3$, $\hat{f}(\xi)$ cannot be recovered uniquely from $f(n)$. We note that the Poisson summation formula gives an equation which relates points in K_2 with points in K_3. In general, given $K\subseteq\widehat{G}$, the Poisson summation formula gives an equation of the form $F(\xi)=\sum_{i=0}^{m-1}\hat{f}(\xi+\eta_i)$, $m\in\mathbb{Z}$ where the only nonvanishing terms have η_i for which $\xi+\eta_i\in K$. We continue with the formal theory.

Lemma 3.2. *Suppose H is a lattice and K a compact subset of $\widehat{G}$. For $\xi\in K$, define $M_\xi=(H^\perp\backslash\{0\})\cap(K-\xi)$. Then the relation $\xi\equiv\xi'\ \Leftrightarrow M_\xi=M_{\xi'}$ is an equivalence relation on K induced by the subgroup H. The equivalence classes are*

$$K_l=\{\xi\in K:\ M_\xi=M_l\},\ l=1,\dots,L. \tag{3.2}$$

Proof. The set $M_\xi=(H^\perp\backslash\{0\})\cap(K-\xi)$ is contained in $H^\perp\cap(K-K)$ where $K-K=\bigcup_{\xi\in K}(K-\xi)$. Since $K-K$ is compact, by Lemma 2.3 in [3], $H^\perp\cap(K-K)$ is a finite set. Since the finite set $H^\perp\cap(K-K)$ has only finitely many subsets, as ξ varies through K, M_ξ will assume only finitely many values, say $M_1,\dots,M_L$. Furthermore, M_ξ must be a finite set for each $\xi\in K$. Thus, we may represet $M_\xi=\{\eta_1,\dots,\eta_{m-1}\}$ where $|M_\xi|=m-1$. Clearly, $\xi\equiv\xi'\ \Leftrightarrow M_\xi=M_{\xi'}$ is an equivalence relation on K with the equivalence classes $K_l=\{\xi\in K:\ M_\xi=M_l\}$, $l=1,\dots,L$. Hence, the set K_l are mutually disjoint and $K=\bigcup_{l=1}^{L}K_l$. Furthermore, by definition of M_l, each K_l consists of the points ξ for which $\xi+\eta\in K$ if $\eta\in M_l\cup\{0\}$, and $\xi+\eta\notin K$ if $\eta\in H^\perp\backslash(M_l\cup\{0\})$. ◻

To facilitate the notation, let

$$M_l = \{\eta_1^{(l)}, \dots, \eta_{m_l-1}^{(l)}\}, \quad l = 1, \dots, L \tag{3.3}$$

be the values assumed by $(H^\perp \setminus \{0\}) \cap (K - \xi)$ as ξ varies through K, and let

$$\tilde{M}_l = M_l \cup \{0\} = \{0 = \eta_0^{(l)}, \dots, \eta_{m_l-1}^{(l)}\}. \tag{3.4}$$

The next lemma explains the relationships between the equivalence classes of the compact set K.

Lemma 3.3. *Suppose H is a lattice and K a compact subset of $\widehat{G}$. Let $\tilde{M}_l$ be defined as in (3.4). Then for each $\eta \in \tilde{M}_l$, there is an $l' \in \{1, \dots, L\}$ such that $K_{l'} = K_l + \eta$, and $\widetilde{M_{l'}} = \tilde{M}_l - \eta$. This yields an equivalence relation on the set of indices $\{1, \dots, L\}$ with the equivalence class of any l consists of $|\tilde{M}_l| = m_l$ elements.*

Proof. First, we note that $(K_l + \eta) \subseteq K$ if $\eta \in \tilde{M}_l$, and $(K_l + \eta) \cap K = \emptyset$ for $\eta \in H^\perp \setminus \tilde{M}_l$. Hence the translate of K_l by an element $\eta \in \tilde{M}_l$ must coincide with an equivalence class $K_{l'}$. Thus, for each $\eta \in \tilde{M}_l$, there exists $l' \in \{1, \dots, L\}$ such that $K_{l'} = K_l + \eta$, and consequently, $\widetilde{M_{l'}} = \tilde{M}_l - \eta$. Now, define a relation on the set of indices $\{1, \dots, L\}$ by letting $l \equiv l'$ if and only if there is $\eta \in \tilde{M}_l$ such that $K_{l'} = K_l + \eta$. The relation is an equivalence relation since
(i) Choosing η to be $0 = \eta_0^{(l)} \in \tilde{M}_l$ implies that $l \equiv l$.
(ii) Assume $l \equiv l'$ is true and choose $\eta' = -\eta \in \tilde{M}_{l'}$ to show that $l' \equiv l$. Note that $-\eta \in \tilde{M}_{l'}$ since $\eta_0^{(l)} - \eta = 0 - \eta = -\eta \in \tilde{M}_{l'}$.
(iii) Assume $l \equiv l'$ and $l' \equiv l''$. Then there exist $\eta \in \tilde{M}_l$ and $\eta' \in \tilde{M}_{l'}$ such that $K_{l'} = K_l + \eta$ and $K_{l''} = K_{l'} + \eta'$ respectively. Hence,

$$K_{l''} = K_l + \eta + \eta' \tag{3.5}$$

To find $\eta_0 \in \tilde{M}_l$ such that $K_{l''} = K_l + \eta_0$, note that $\tilde{M}_{l'} = \tilde{M}_l - \eta$ implies that there exists $\eta_0 \in \tilde{M}_l$ such that $\eta' = \eta_0 - \eta$. Now (3.5) becomes $K_l + \eta + \eta' = K_l + \eta + (\eta_0 - \eta) = K_l + \eta_0$. □

Convention 3.4. Let J be the number of equivalence classes for the set $S = \{1, \dots, L\}$ where $1 \le J \le L$. Assume that each equivalence class S_j for $j = 1, \dots, J$ contains n_j elements. We rearrange the sets K_l so that the sets which belong to the same equivalence class are grouped together. This gives

$$K_1, \dots, K_L = \underbrace{K_1, \dots, K_{n_1}}_{j=1}, \underbrace{K_{n_1+1}, \dots, K_{n_1+n_2}}_{j=2}, \dots, \underbrace{K_{n_1+\dots+n_{J-1}+1}, \dots, K_{n_1+\dots+n_J}}_{j=J}.$$

Note that for each $j \in \{1, \dots, J\}$,

$$l = (\sum_{j'=1}^{j-1} n_{j'}) + 1 + k, \qquad k = 0, \dots, n_j - 1. \tag{3.6}$$

We may describe this arrangement by a double index (j, k), where j identifies the equivalence class and k is as in (3.6). Thus using this correspondence between l and (j, k), we write $K_{j,k}$ for K_l, $\eta_n^{(j,k)}$ for $\eta_n^{(l)}$ and $\widetilde{M}_{(j,k)} = \{0 = \eta_0^{(j,k)}, \dots, \eta_{n_j-1}^{(j,k)}\}$ for $\widetilde{M}_l = \{0 = \eta_0^{(l)}, \dots, \eta_{m_l-1}^{(l)}\}$. Furthermore, one can order the sets K_l in each equivalence class in such a way that $K_{j,k} = K_{j,0} + \eta_k^{(j,0)}$ and $\widetilde{M}_{j,k} = \widetilde{M}_{j,0} - \eta_k^{(j,0)}$, $j = 1, \dots, J$, $k = 0, \dots, n_j - 1$.

To clarify the abstract notation, we continue by giving an example in one-dimension. For an example in two-dimensions, see [2] and the references given there.

Example 3.5. : Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}, \widehat{G} = \mathbb{Z}, H = \{j/N : j = 0, \ldots, N-1\}$ with addition modulo 1 where $N \geq 5$ is a positive integer. It follows that $H^\perp = N\mathbb{Z}$. Let $K = \{-P, \ldots, N+P\}$ where $P < \frac{N}{2} - 1$ is a positive integer. Then for $\xi \in K$, we have $(H^\perp \setminus \{0\}) \cap (K - K) = \{-N, N\}$. It follows that K_1, K_2, and K_3 are the sets $\{P+1, \ldots, N-P-1\}, \{-P, \ldots, P\}$, and $\{N-P, \ldots, N+P\}$ respectively. Hence, $L = 3$ and M_l for $l = 1, 2$ and 3 is respectively $\{\emptyset\}$, $\{N\}$, and $\{-N\}$. The equivalence relation on the set of indices gives the double indices as follows: $S_1 = \{1\}$ with $\eta_0^{(1,0)} = 0$, $S_2 = \{2, 3\}$ with $\eta_1^{(2,0)} = N$ and $\eta_1^{(2,1)} = -N$. Furthermore, the compact sets can be reindexed and defined as $K_1 = K_{1,0}$, $K_2 = K_{2,0}$ and $K_3 = K_{2,1}$ where $K_{2,1} = K_{2,0} + \eta_1^{(2,0)}$.

The following theorem is the main result of our paper.

Theorem 3.6. *: Suppose H is an admissible lattice and R a fundamental domain of $H^\perp$. Assume that $f \in L_2(G)$ is continuous, f vanishes on H, and that $\hat{f}$ vanishes almost everywhere outside a compact set $K \subset \widehat{G}$. Let $K_{j,k}$ be the sets of the decomposition of K according to Convention 3.4. Then there exist continuous functions $h_{j,k} \in L_2(G)$ with $\widehat{h}_{j,k}$ vanishing almost everywhere outside $K_{j,k}$ for $j = 1, \ldots, J$, $k = 1, \ldots, n_j - 1$ such that*

$$\widehat{f}(\xi) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi) - \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) \tag{3.7}$$

for almost every $\xi \in \widehat{G}$ and

$$f(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} h_{j,k}(x)(1 - e^{-2\pi i \langle x, \eta_k^{(j,0)} \rangle}) \tag{3.8}$$

for almost every $x \in G$.

Proof. Let $h_{j,k} \in L_2(G)$ be a continuous function such that $\widehat{h}_{j,k}(\xi) = \chi_{K_{j,k}}(\xi)\, \widehat{f}(\xi)$ for $j = 1, \ldots, J$ and $k = 0, \ldots, n_j - 1$. Then

$$\widehat{f}(\xi) = \sum_{j=1}^{J} \sum_{k=0}^{n_j-1} \widehat{h}_{j,k}(\xi) \qquad a.e. \tag{3.9}$$

Since $\hat{f} \in L_1(\widehat{G})$ and $\int_H |f(y)|^2 \, dm_H(y) = 0$ by hypothesis, the Poisson summation formula (2.3) can be applied. Thus, we have

$$\sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta) = \frac{1}{m_{\widehat{G}}(R)} \sum_{y \in H} f(y) e^{-2\pi i \langle y, \xi \rangle} \qquad a.e. \tag{3.10}$$

where (3.10) converges in $L_2(\widehat{G}/H^\perp)$. Since f vanishes on H, the right hand side of (3.10) is equal to zero and hence for almost every $\xi \in K_{j,0}$

$$\sum_{\eta \in H^\perp} \widehat{f}(\xi + \eta) = \sum_{k=0}^{n_j-1} \widehat{f}(\xi + \eta_k^{(j,0)}) = 0 \tag{3.11}$$

where $\eta_0^{(j,0)} = 0$ and $j = 1, \dots, J$. Combining (3.9) and (3.11) gives

$$\widehat{h}_{j,0}(\xi) + \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) = 0 \tag{3.12}$$

for almost every $\xi \in K_{j,0}$. If $\xi \notin K_{j,0}$, then $\xi + \eta_k^{(j,0)} \notin K_{j,k}$ for fixed j and $k = 1, \dots, n_j - 1$. Hence, (3.12) holds for almost every $\xi \in \widehat{G}$. We solve for $\widehat{h}_{j,0}$ in (3.12) and substitute in (3.9) to get

$$\begin{aligned} \widehat{f}(\xi) &= \sum_{j=1}^{J} \Big(- \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) + \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi) \Big) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} \widehat{h}_{j,k}(\xi) - \widehat{h}_{j,k}(\xi + \eta_k^{(j,0)}) \end{aligned} \tag{3.13}$$

for almost every $\xi \in \widehat{G}$. The inverse Fourier transform of (3.13) gives (3.8). □

Note that if $K + \eta, \ \eta \in H^{\perp}$ are disjoint, then $J = 1, \ n_1 = 1$ so (3.8) reads $f(x) = 0$ which is consistent with the classical sampling theorem.

The next corollary considers the case that f does not vanish on H, but it vanishes on a coset $\alpha + H$ of H.

Corollary 3.7. *: Assume that the hypotheses of the Theorem 3.6 holds except the condition that f vanishes on H. Assume instead that f vanishes on $\alpha + H$, where $\alpha \in G$.*
Then there exist functions $h_{j,k} \in L_2(G)$ with $\widehat{h}_{j,k}$ vanishing almost everywhere outside $K_{j,k}$ for $j = 1, \dots, J, \ k = 1, \dots, n_j - 1$ such that

$$f(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} h_{j,k}(x)(1 - e^{-2\pi i \langle x-\alpha, \eta_k^{(j,0)} \rangle}) \tag{3.14}$$

for almost every $x \in G$.

Proof. Define $g(x) = f(x + \alpha)$. Note that $\widehat{g}$ vanishes almost everywhere outside K and g vanishes on H. Thus Theorem 3.6 can be applied to g to give

$$g(x) = \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} \widetilde{h}_{j,k}(x)(1 - e^{-2\pi i \langle x, \eta_k^{(j,0)} \rangle}).$$

Define $h_{j,k}(x) = \widetilde{h}_{j,k}(x - \alpha)$. Then

$$\begin{aligned} f(x) &= g(x - \alpha) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} \widetilde{h}_{j,k}(x - \alpha)(1 - e^{-2\pi i \langle x-\alpha, \eta_k^{(j,0)} \rangle}) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{n_j-1} h_{j,k}(x)(1 - e^{-2\pi i \langle x-\alpha, \eta_k^{(j,0)} \rangle}). \end{aligned}$$

This completes the proof. □

We conclude the paper with the following example which is an important case considered in [1].

Example 3.8. : Assume H is an admissible lattice with R a fundamental domain of $H^\perp$. Suppose K' is a nonempty compact subset of R. Let $K = R \cup (\eta_1 + K')$ where $\eta \in H^\perp \setminus \{0\}$. Then the decomposition of K yields

$$K_1 = K' + \eta,\ K_2 = K' \quad \text{and} \quad K_3 = R \setminus K'.$$

We obtain two equivalence classes:

$$K_{1,0} = \eta + K',\ K_{1,1} = K' \quad \text{and} \quad K_{2,0} = R \setminus K'.$$

Hence $n_1 = 2,\ n_2 = 1$ and $\eta_1^{(1,0)} = -\eta$. Equation (3.14) implies that

$$f(x) = h(x)(1 - e^{2\pi i \langle x-\alpha, \eta\rangle}) \tag{3.15}$$

with $\widehat{h}$ vanishing almost everywhere outside $K_{1,1} = K'$.

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