Homework Week 6 Solutions

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#11a. Proof by induction on $n$.

**Base Case** ($n = 1$): $\text{LHS} = l_1 = 1$ \hspace{1cm} $\text{RHS} = f_0 + f_2 = 0 + 2 = 1$

**Induction Hypothesis**: Assume $l_k = f_{k-1} + f_{k+1}$ for all $k < n$.

**Induction Step**: Prove this is true for $n$, i.e. $l_n = f_{n-1} + f_{n+1}$.

$LHS = l_n = l_{n-1} + l_{n-2} = (f_{n-2} + f_n) + (f_{n-3} + f_{n-1})$ by the induction hypothesis. Rearranging terms, we get:

$$l_n = (f_{n-2} + f_{n-3}) + (f_n + f_{n-1}) = f_{n-1} + f_{n+1}$$

by the Fibonacci recurrence relation.

#11b. Proof by induction on $n$.

**Base Case** ($n = 0$): $\text{LHS} = l_0^2 = 2^2 = 4$ \hspace{1cm} $\text{RHS} = l_0 l_1 + 2 = 2 \cdot 1 + 2 = 4$

**Induction Hypothesis**: Assume $l_0^2 + l_1^2 + \cdots + l_k^2 = l_k l_{k+1} + 2$ for some $k \geq 0$.

**Induction Step**: Prove this is true for $k + 1$, i.e. $l_0^2 + l_1^2 + \cdots + l_k^2 + l_{k+1}^2 = l_{k+1} l_{k+2} + 2$

$LHS = l_0^2 + l_1^2 + \cdots + l_k^2 + l_{k+1}^2 = (l_k l_{k+1} + 2) + l_{k+1}^2$ by the induction hypothesis. Rearranging terms, we get:

$$l_0^2 + l_1^2 + \cdots + l_k^2 + l_{k+1}^2 = l_{k+1} (l_k + l_{k+1}) + 2 = l_{k+1} l_{k+2} + 2$$

by the Lucas recurrence relation.

#16: Suppose you want to make fruit salad and you have apples, bananas, clementines and Dole pineapples. If you have to have at most two apples, at most 6 and an even number of bananas, an even number of clementines and at least one Dole pineapple, how many ways can you make a fruit salas using $n$ ingredients? The answer is the coefficient of $x^n$ in the given product of polynomials.

#17: $h_n$ is the coefficient of $x^n$ in

$$H(x) = (1 + x^2 + x^4 + \cdots)(1 + x + x^2)(1 + x^3 + x^6 + \cdots)(1 + x) = \frac{1}{1 - x^2} (1 + x + x^2) \frac{1}{1 - x^3} (1 + x)$$

$$= \frac{1}{(1 - x)(1 + x)} (1 + x + x^2)^2 \frac{1}{(1 - x)} (1 + x + x^2) (1 + x) = \frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} (n + 1) x^n$$

Therefore $h_n = (n + 1)$

#22: Let $a_n = n!$ be our sequence. Its exponential generating function is

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$
\#25: \( H(x) = \left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \left( 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \right)^2 \)

\[ = \frac{1}{2}(e^x + e^{-x}) \left[ \frac{1}{2}(e^x - e^{-x}) e^{2x} \right] = \frac{1}{4} (e^{4x} - 1) = \frac{1}{4} \left( \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - 1 \right) = \sum_{n=1}^{\infty} \frac{4^n x^n}{4^n n!} - \frac{1}{4} (x^0 - 1) = \sum_{n=1}^{\infty} \frac{4^n x^n}{4^n n!} \]

Therefore \( h_n = 4^{n-1} \) for \( n \geq 1 \) and \( h_0 = 0 \).

\#27: \( H(x) = \left( \frac{1}{2}(e^x + e^{-x}) - 1 \right) \left( \frac{1}{2}(e^x + e^{-x}) - 1 \right) e^x e^x e^x \) for the digits 1, 3, 5, 7, 9 respectively.

\[ H(x) = \left[ \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - e^x - e^{-x} + 1 \right] e^{3x} = \frac{1}{4} e^{5x} - e^{4x} + \frac{3}{4} e^{3x} - e^{2x} + \frac{1}{4} e^{x} \]

\[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \frac{5^n}{4} - 4^n + \frac{3^{n+1}}{2} - 2^n + \frac{1}{4} \right) \]

Therefore \( h_n = \frac{5^n}{4} - 4^n + \frac{3^{n+1}}{2} - 2^n + \frac{1}{4} \)

(Note \( h_0 = h_1 = h_2 = h_3 = 0 \) and \( h_4 = 6 \) which is correct!)