Homework Week 4 Solutions

pp. 154−160 :

#6 The coefficient of $x^5y^{13}$ is $\left(\begin{array}{c}18 \\ 5 \end{array}\right)3^5 \cdot (-2)^{13}$. The coefficient of $x^8y^9$ is 0 since the degree of the polynomial is 18 and the degree of $x^8y^9$ is 17.

#11 Let $a, b, c$ be special elements of an $n$-set (so there are $n − 3$ “other” elements in the set). The $\binom{n}{k}$ on the LHS represents the number of ways of choosing $k$ elements from the entire $n$-set. The $-\binom{n − 3}{k}$ on the LHS represents getting rid of the subsets of size $k$ consisting of only the “other” elements. Thus, the LHS is the number of ways of choosing a subset of size $k$, with at least one of $a, b, c$.

Note a subset consisting of at least one of $a, b, c$ either

(i) has $a$ in it $\binom{n-1}{k-1}$ ways to choose the other $k-1$ elements besides $a$ or

(ii) $b$, but not $a$ $\binom{n-2}{k-1}$ ways to choose the other $k-1$ elements since we have $b$, not $a$ and only have $n-2$ elements left to choose from or

(iii) $c$ but neither $a$ nor $b$ $\binom{n-3}{k-1}$ ways to choose the other $k-1$ elements since we have $c$, not $a$ or $b$ and only have $n-3$ elements left to choose from. This is the RHS!

#14: RHS = $r \frac{r}{r-k} \binom{r-1}{k} = \frac{r(r-1)!}{(r-k)!} = \frac{r!}{k!(r-k)!} = \binom{r}{k} = $LHS

Combinatorial Proof: We'll actually show $\binom{r}{k} \binom{r}{k} = r \binom{r-1}{k}$.

If we want to choose one special element and $k$ “okay” elements from an $r$-set, we can either

(i) choose the special element in $r$ ways and the $k$ “okay” elements from the rest of the set in $\binom{r-1}{k}$ ways (This is the RHS) or

(ii) choose the “okay” elements first in $\binom{r}{k}$ ways and the the special element from the remaining $r-k$ elements. (This is the LHS.)

#15:

$$\sum_{k=1}^{n}(-1)^{n-1}k\binom{n}{k} = \sum_{k=1}^{n}(-1)^{n-1}k\frac{n!}{k!(n-k)!} = \sum_{k=1}^{n}(-1)^{n-1}\frac{n(n-1)!}{(k-1)!(n-k)!}$$

$$= n\sum_{k=1}^{n}(-1)^{n-1}\frac{(n-1)!}{(k-1)!(n-k)!} = n\sum_{m=0}^{n-1}(-1)^{n-1}\frac{(n-1)!}{m!(n-1-m)!} = n\sum_{m=0}^{n-1}(-1)^{n-1}\binom{n-1}{m}$$

where we substituted $m = k - 1$ into the equation in $t$

If $n = 0$, this sum is clearly 0. If $n > 0$, then the sum is zero by Theorem 3 from class (aka 5.4 on page 133).

#40: $\binom{9}{3,3,1,2} \cdot (-1)^3 \cdot 2 \cdot (-2)^2 = \frac{9!}{3!3!2!} \cdot (-8) = -40320$
#2: Let $P_k$ be the property that a number is divisible by $k$ and $A_k$ be the set of numbers with property $P_k$ between 1 and 10,000 inclusive.

$|A_4| = 2500, \ |A_6| = \left\lfloor \frac{10000}{6} \right\rfloor = 1666, \ |A_7| = \left\lfloor \frac{10000}{7} \right\rfloor = 1428, \ |A_{10}| = 1000$

Note that $|A_4 \cap A_6| = |A_{12}| = \left\lfloor \frac{10000}{12} \right\rfloor$ because 12 is the number that is a multiple of 4 and 6, i.e., LCM(4,6)=12.

$|A_4 \cap A_6| = |A_{12}| = \left\lfloor \frac{10000}{12} \right\rfloor = 833, \ |A_4 \cap A_7| = |A_{28}| = \left\lfloor \frac{10000}{28} \right\rfloor = 357, \ |A_4 \cap A_{10}| = |A_{40}| = \left\lfloor \frac{10000}{40} \right\rfloor = 250, \ |A_6 \cap A_7| = |A_{42}| = \left\lfloor \frac{10000}{42} \right\rfloor = 238$

$|A_4 \cap A_6 \cap A_7| = |A_{84}| = \left\lfloor \frac{10000}{84} \right\rfloor = 119, \ |A_4 \cap A_6 \cap A_{10}| = |A_{60}| = \left\lfloor \frac{10000}{60} \right\rfloor = 166, \ |A_4 \cap A_7 \cap A_{10}| = |A_{140}| = \left\lfloor \frac{10000}{140} \right\rfloor = 71, \ |A_6 \cap A_7 \cap A_{10}| = |A_{210}| = \left\lfloor \frac{10000}{210} \right\rfloor = 47$

$|A_4 \cap A_6 \cap A_7 \cap A_{10}| = |A_{420}| = \left\lfloor \frac{10000}{420} \right\rfloor = 23.$

$|A_4 \cap A_6 \cap A_7 \cap A_{10}| = 10000 - (|A_4| + |A_6| + |A_7| + |A_{10}|) + (|A_{12}| + |A_4 \cap A_7| + |A_4 \cap A_{10}| + |A_6 \cap A_7| + |A_6 \cap A_{10}| + |A_{12}|) - (|A_4 \cap A_{12}| + |A_4 \cap A_6 \cap A_{12}| + |A_4 \cap A_7 \cap A_{12}| + |A_6 \cap A_7 \cap A_{12}| + |A_7 \cap A_{12}|) = 5429$

#6: The box either has 0, 1, 2 or 3 plain doughnuts.

0 plain doughnuts: 1 way (6 chocolate and 6 cinnamon)
1 plain doughnut: 2 ways (6 chocolate and 5 cinnamon OR 5 chocolate and 6 cinnamon)
2 plain doughnuts: 3 ways (6 chocolate and 4 cinnamon OR 5 chocolate and 5 cinnamon OR 4 chocolate and 6 cinnamon)
3 plain doughnuts: 4 ways (6 chocolate and 3 cinnamon OR 5 chocolate and 4 cinnamon OR 4 chocolate and 5 cinnamon OR 3 chocolate and 6 cinnamon)